Slow dynamics and fluctuations in quasi-integrable systems

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Outline

- **Introduction and Motivation**
  - Integrable systems with external noise
- **Slow dynamics in FPU(T) chain**
  - Introduction to the FPU problem
  - Weakly conserved quantities and GGE ensemble
- **Fluctuation theorem**
- **Outlook: solar system dynamics**
- **Summary**

Fluctuation Theorem → arXiv:1807.08497
Slow Dynamics → In preparation

Introduction- Integrable systems

*I will consider only *isolated, classical systems*.

- **<u>Integrable system:</u>** *Integrable system* of size N has N integrals of motion

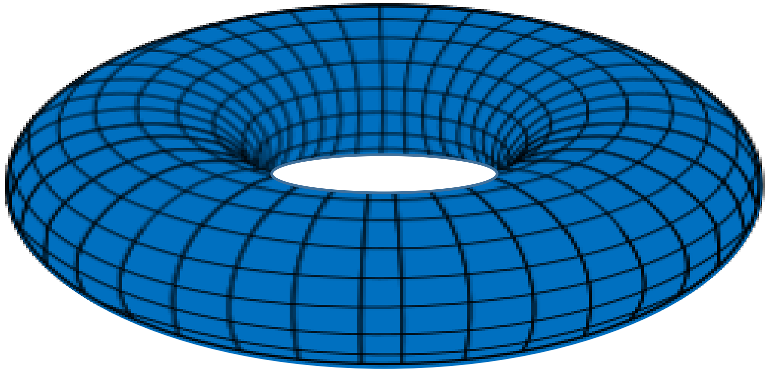
Hamiltonian $\mathcal{H}(\vec{q},\vec{p})$

There is a canonical transformation:

$$(\vec{q},\vec{p}) \to (\vec{\theta},\vec{I}) \qquad \{I_j, I_k\} = 0$$

$$\mathcal{H}(\vec{q},\vec{p}) \to \mathcal{H}(\vec{I})$$

If the frequencies are incommensurate–
the tori are filled



$$\dot{I}_k = -\frac{\partial \mathcal{H}}{\partial \theta_k} = 0 \qquad \Longrightarrow \qquad I_k = \text{Constant}$$

$$\dot{\theta}_k = \frac{\partial \mathcal{H}}{\partial I_k} = \Omega_k \qquad \qquad \theta_k = \Omega_k t + \theta_k(0)$$

$\Longrightarrow$ Flow is laminar along N-dimensional torus within the $2N$-dimensional phase space.

- **<u>Quasi-Integrable system:</u>** *Slightly perturbed* integrable system

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_p, \qquad \mathcal{H}_0 \text{ is integrable, and } \mathcal{H}_p \ll \mathcal{H}_0$$

Motivation– Integrable systems perturbed by an external noise

Nguyen Thu Lam & Kurchan (2014)

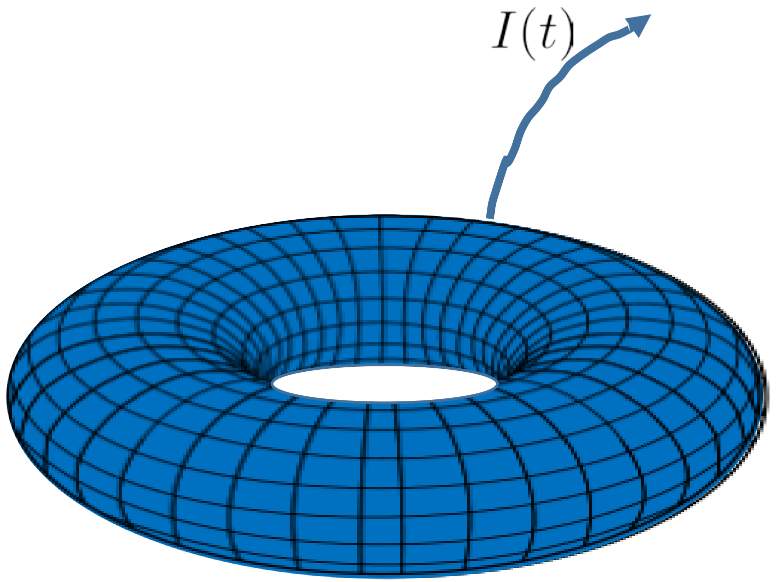
What happens if an integrable system *is slightly perturbed by noise?*

Hamiltonian $\mathcal{H}_0(\vec{q},\vec{p}) + \epsilon^{1/2}\mathcal{H}_p(\vec{q},\vec{p})$

$\dot{I}_k = \epsilon^{1/2}\zeta_k$

$\dot{\theta}_k = \Omega_k + \epsilon^{1/2}\eta_k$

η_k, ζ_k (multipicative) noise



$\Delta x(t) \propto e^{\lambda t}$

<u>Two important conclusions:</u>

$\Longrightarrow$

(1) Chaos (Lyapunov separation) is preliminary restricted on the tori.

(2) Suggests a hydrodynamic description for diffusion/drift of the action variables.

Introduction

*The case of a stochastic perturbation was given as a motivation/intuition.

We will look at *deterministic many-body dynamics*

(arguably they should behave the same as the stochastic ones)

- **The FPU problem (Fermi, Pasta, Ulam, and Tsingou 1955)**

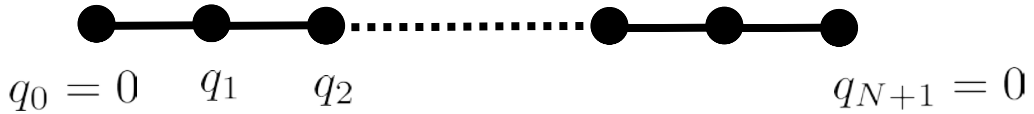
- **Solar system dynamics (Laplace and Lagrange ~1780)**

— We offer a new perspective to look at the geometry of quasi-integrable systems in general.
— This opens a window to explore the system, e.g., its slow dynamics or fluctuations, more efficiently.
— Connects with other quasi-integrable systems: quantum systems, weak turbulence.

FPU Chain- introduction

(Fermi—Pasta—Ulam—Tsingou)

Isolated, finite chain with fixed ends



$$\mathcal{H}(\vec{q},\vec{p}) = \sum_{n=1}^{N} \frac{p_n^2}{2} + \sum_{n=0}^{N} V_{\text{FPU}}(q_{n+1} - q_n)$$

$$V_{\text{FPU}}(r) = \frac{r^2}{2} + \alpha\frac{r^3}{3} + \beta\frac{r^4}{4}$$

The FPU Problem/Paradox

small energy $\epsilon \equiv E/N \ll 1$

$Q_k, P_k \equiv$ Fourier (normal) modes of the linear chain.

$$\begin{pmatrix} Q_k \\ P_k \end{pmatrix} = \sqrt{\frac{2}{N+1}} \sum_{i=1}^{N} \begin{pmatrix} q_i \\ p_i \end{pmatrix} \sin\left(\frac{\pi k i}{N+1}\right)$$

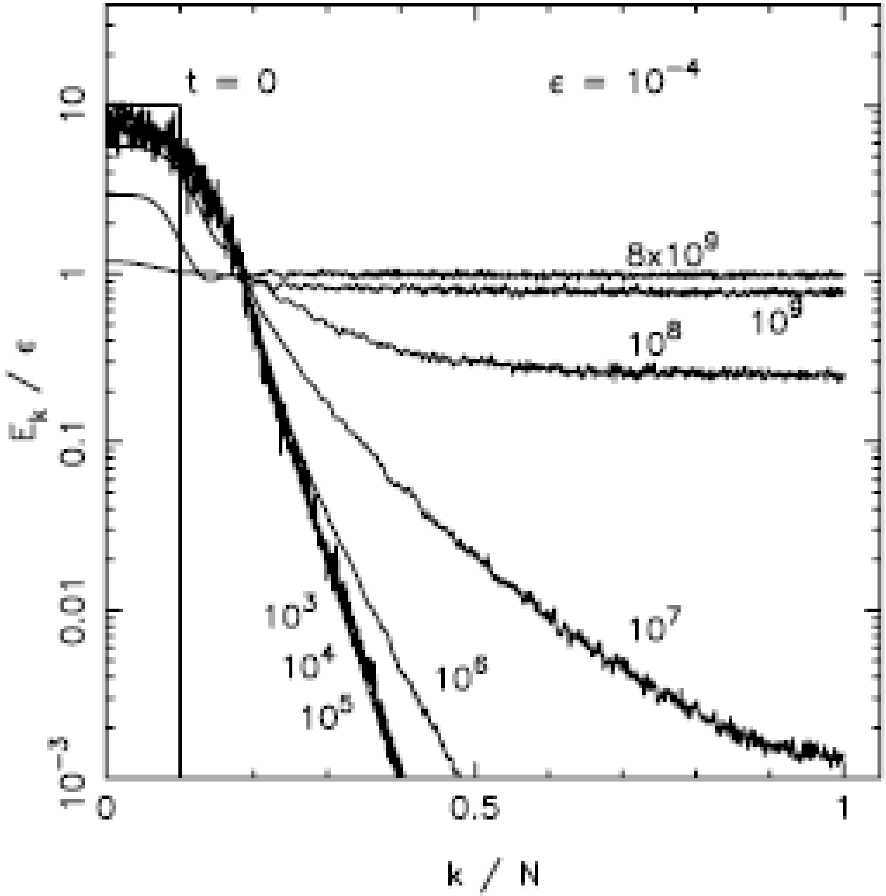
Look at: $E_k = \frac{1}{2}(P_k^2 + \omega_k^2 Q_k^2), \quad w_k = 2\sin\left(\frac{\pi k}{2N+2}\right)$

Expect that: $\lim_{t\to\infty} \frac{1}{t}\int_0^t E_k = \epsilon$

Initial condition: only fraction of the low modes are excited, e.g., $0 < k/N < 0.1$.

FPU Chain- introduction

$N = 1023,\ \epsilon \equiv E/N = 10^{-4},\ \alpha = 1,\ \beta = 2$



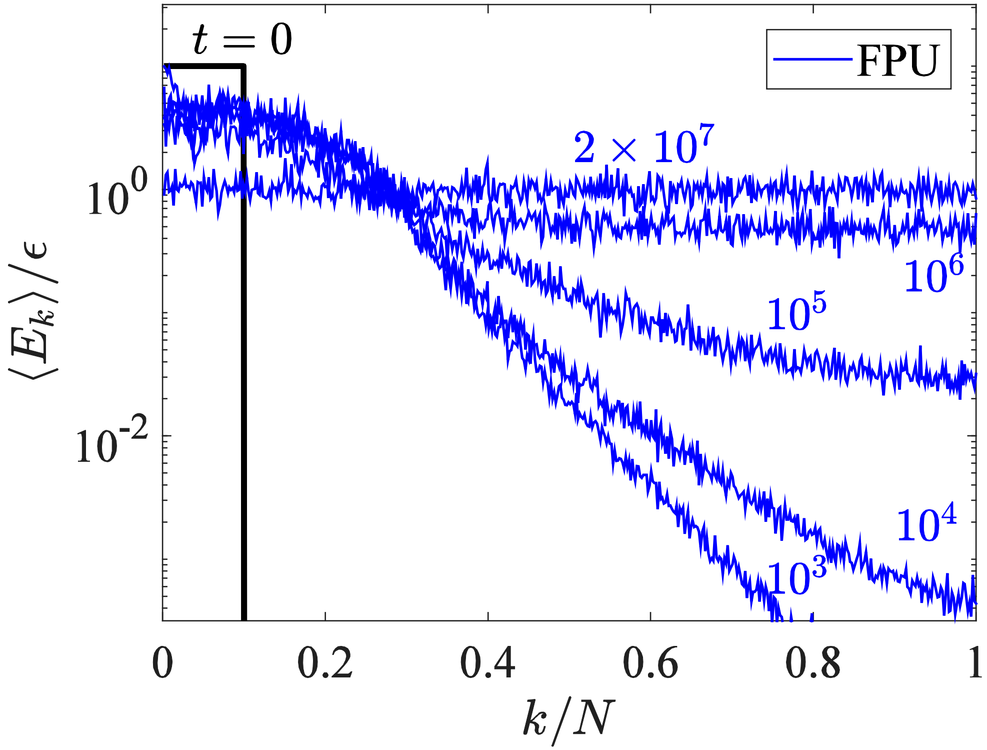
time average

$$\bar{E}_k(t) = \frac{1}{t}\int_0^t E_k(t')dt'$$

Benettin et. al. (2013)

$N = 511,\ \epsilon \equiv E/N = 10^{-3},\ \alpha = 1,\ \beta = 2$

average over 68 ϕ_k



Initial ensemble: ensemble of different ϕ_k

$$\begin{pmatrix} \omega_k Q_k \\ P_k \end{pmatrix} \to \begin{pmatrix} \cos\phi_k & -\sin\phi_k \\ \sin\phi_k & \cos\phi_k \end{pmatrix} \begin{pmatrix} \omega_k Q_k \\ P_k \end{pmatrix}$$

$$E_k = \frac{1}{2}(P_k^2 + \omega_k^2 Q_k^2)$$

FPU Chain- introduction

Two time-scales:

- Fast– energy is shared between fraction of the normal modes.
- Slow– equipartition between all modes.

The short time-scale is intimately related to the Toda chain (M. Toda 1970).

Zabusky & Kruskal (1965); Ferguson et. al. (1982); Benettin et. al. (2013)

The Toda chain is integrable!

$$V_{\mathrm{Toda}}(r) = V_0(e^{\lambda r} - \lambda r - 1)$$

$$\Big\downarrow \text{ choosing } \lambda = 2\alpha,\ V_0 = \lambda^{-2}$$

$$\boxed{V_{\mathrm{FPU}} = \underbrace{V_{\mathrm{Toda}}}_{\text{integrable}} - \overbrace{\left(\frac{2}{3}\alpha^2 - \beta\right)\frac{r^4}{4} + \frac{\alpha^3}{3}\frac{r^5}{5} + \dots}^{\text{perturbation}}}$$

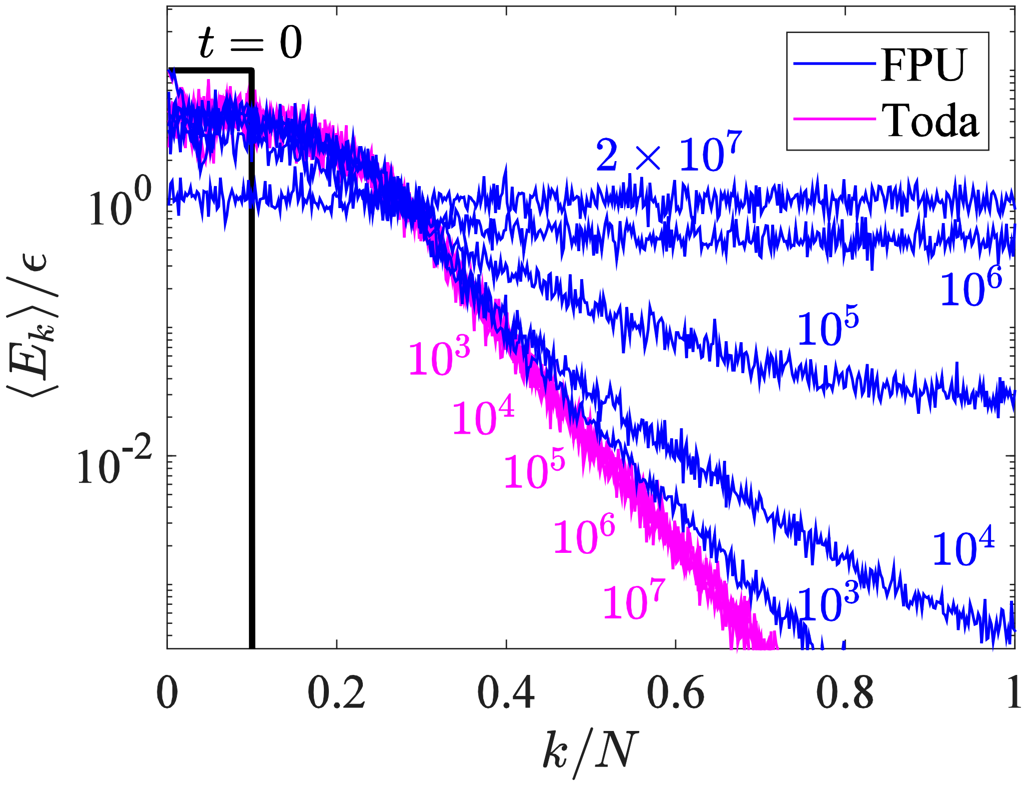
$$V_{\mathrm{FPU}}(r) = \frac{r^2}{2} + \alpha\frac{r^3}{3} + \beta\frac{r^4}{4}$$

$$V_{\mathrm{Linear}}(r) = \frac{r^2}{2}$$

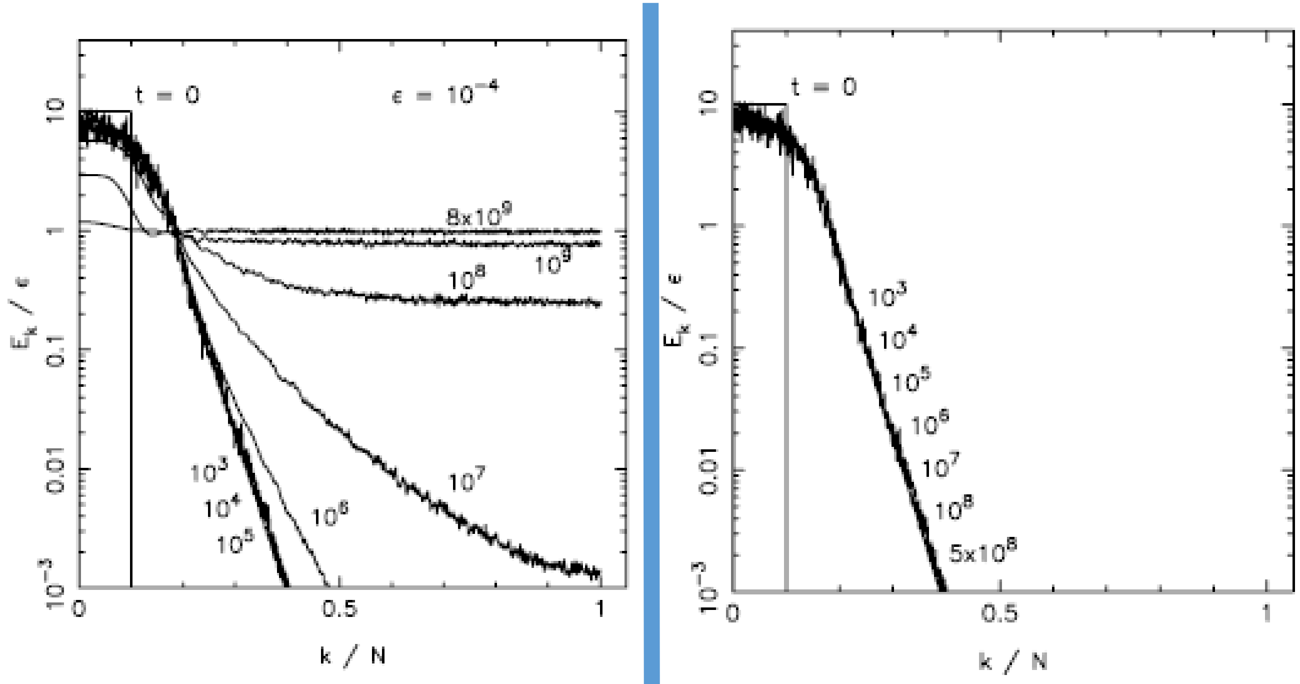
$$V_{\mathrm{Toda}}(r) = V_0(e^{\lambda r} - \lambda r - 1)$$

FPU Chain- introduction

$$N = 511,\ \epsilon \equiv E/N = 10^{-3},\ \alpha = 1,\ \beta = 2$$



The fast time-scale of FPU is the time it takes to fill the Toda tori.



Benettin et. al. (2013)

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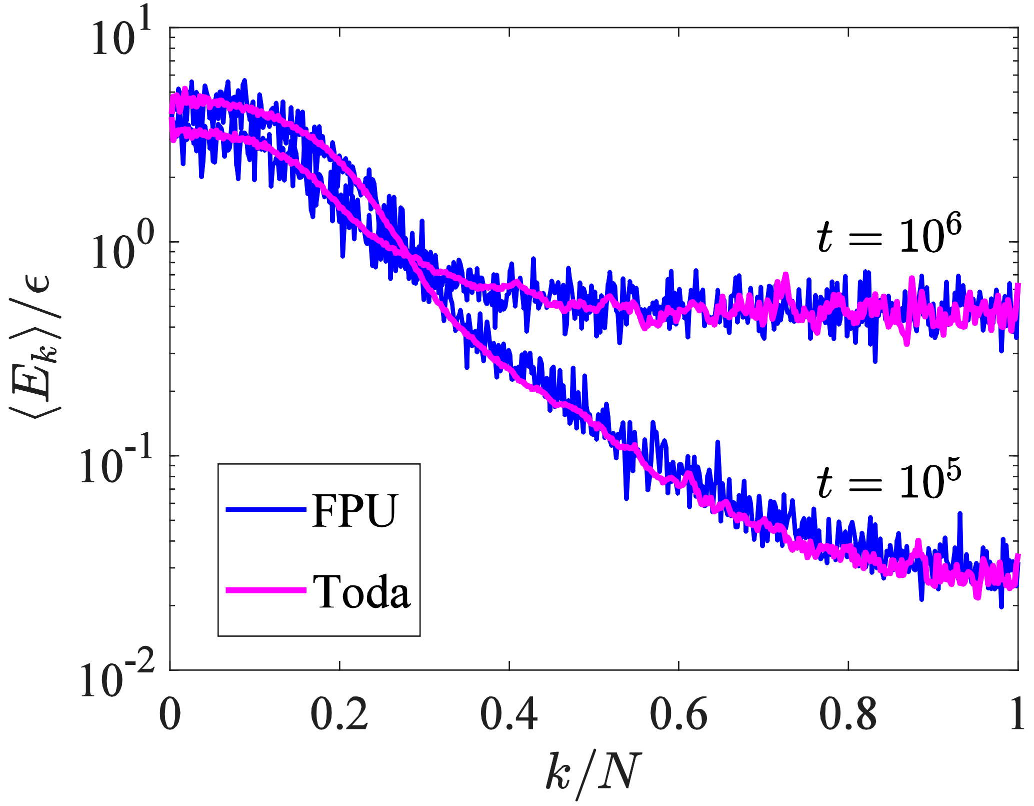
The Slow Dynamics of FPU

Main Observation:

In the thermodynamic (large N) limit, the FPU dynamics drifts slowly between metastable states, each of which is characterized by Toda tori.

➡ The slow thermalization of the FPU chain is dictated by the dynamics of the Toda conserved variables



magenta: Toda dynamics starting from the blue curve and integrated for $\Delta t = 10^6$.

How can we characterize the ensemble of a quasi-static state?

The Slow Dynamics of FPU- Generalized Gibbs Ensemble (GGE)

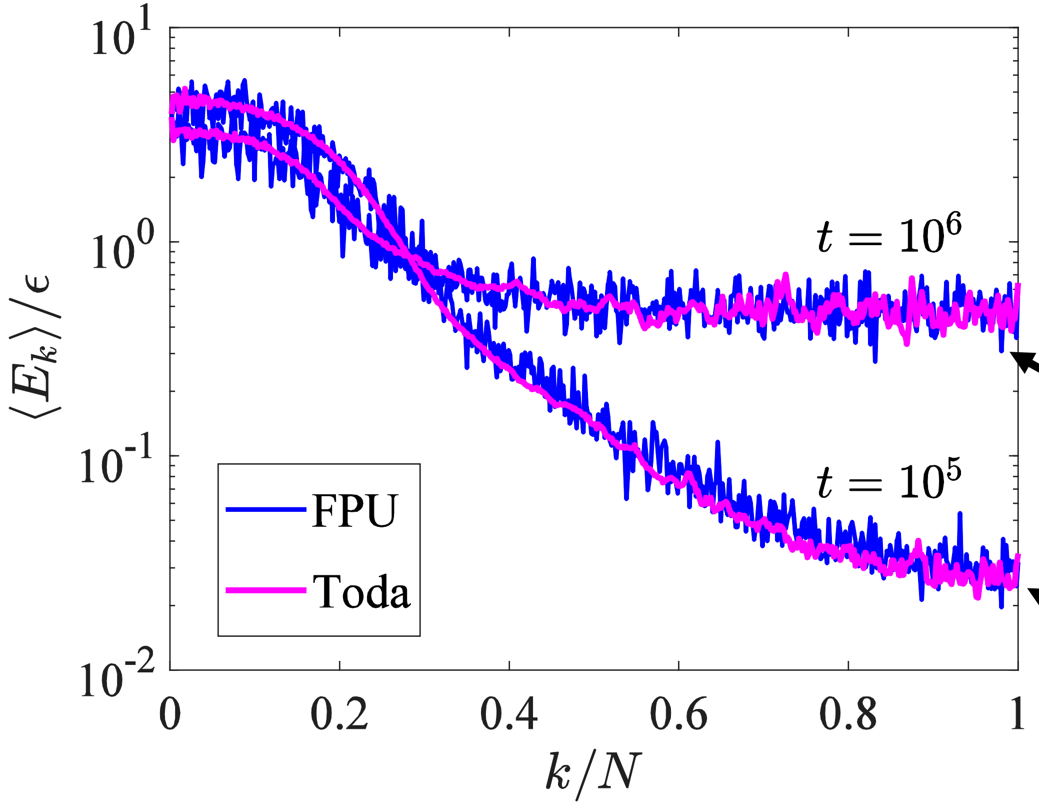
Gibbs ensemble generalized Gibbs ensemble

$$e^{-\beta E-\mu N} \to e^{-\sum_i \beta_i J_i}$$

$\{J_i\}$ set of conserved quantities

$\{\beta_i\}$ Lagrange multipliers

Principle of maximum entropy (Jaynes 1957)
In quantum systems, e.g., Rigol et. al. (2007)



<u>magenta</u>: Toda dynamics starting from the blue curve and integrated for $\Delta t = 10^6$.

$e^{-\sum \beta_i(t) J_i(t)}$

$e^{-\sum \beta_i(t) J_i(t)}$

$\{J_i\}$ conserved quantities of the Toda chain

The Slow Dynamics of FPU- conserved quantities of Toda

The choice of conserved quantities:

- The action variables of Toda are not easy to compute.
- The Hamiltonian dynamics of Toda is equivalent to the Lax equation:

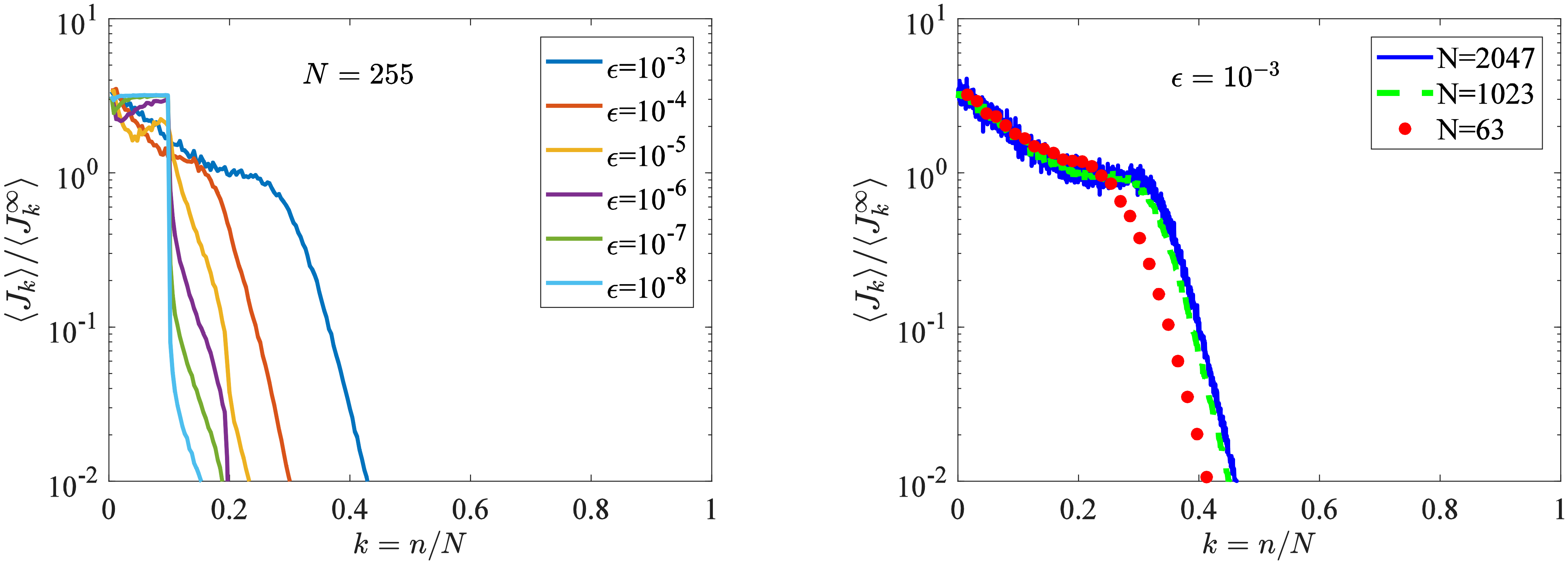
$$\begin{aligned} \dot{q}_k &= \frac{\partial \mathcal{H}}{\partial p_k} \\ \dot{p}_k &= -\frac{\partial \mathcal{H}}{\partial q_k} \end{aligned} \qquad \Longleftrightarrow \qquad \dot{L} = B \cdot L - L \cdot B \implies \text{the eigenvalues of } L(\vec{q}\,,\vec{p}\,) \text{ do not vary in time.}$$

Hénon (1974); Flaschka (1974)

- *We took the gaps of the eigenvalues as our conserved quantities:*
  - They are correlated with the action variables: the phase space volume is given by the number of non-vanishing gaps.
  - They are correlated with the energy of the normal (Fourier) modes.

Ferguson et. al. (1982)

The Slow Dynamics of FPU- conserved quantities of Toda



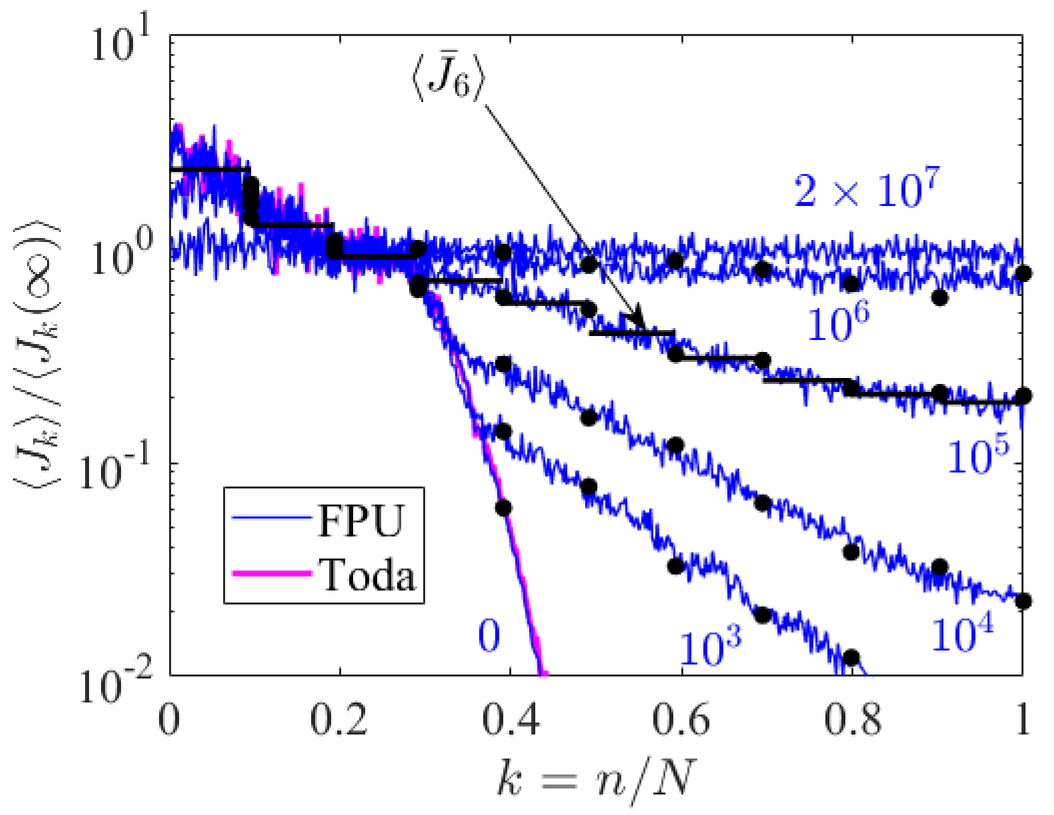
$$E_k = \begin{cases} 10\epsilon, & 0 < k/N \leq 0.1 \\ 0, & 0.1 < k/N \leq 1 \end{cases}$$

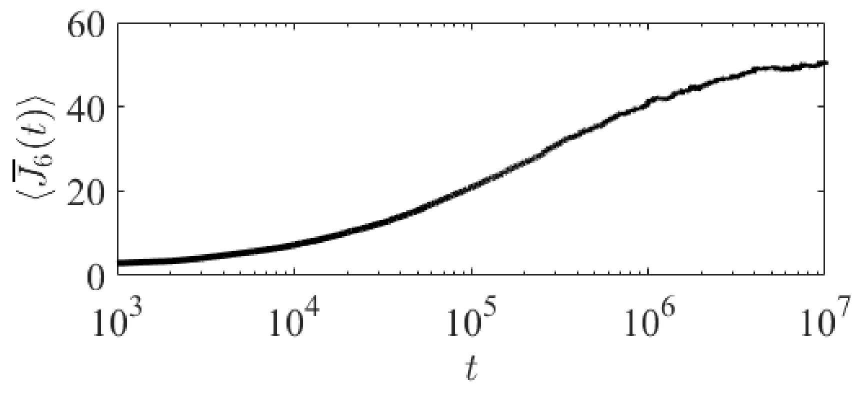
The Slow Dynamics of FPU- conserved quantities of Toda

- *We preform coarse-graining*

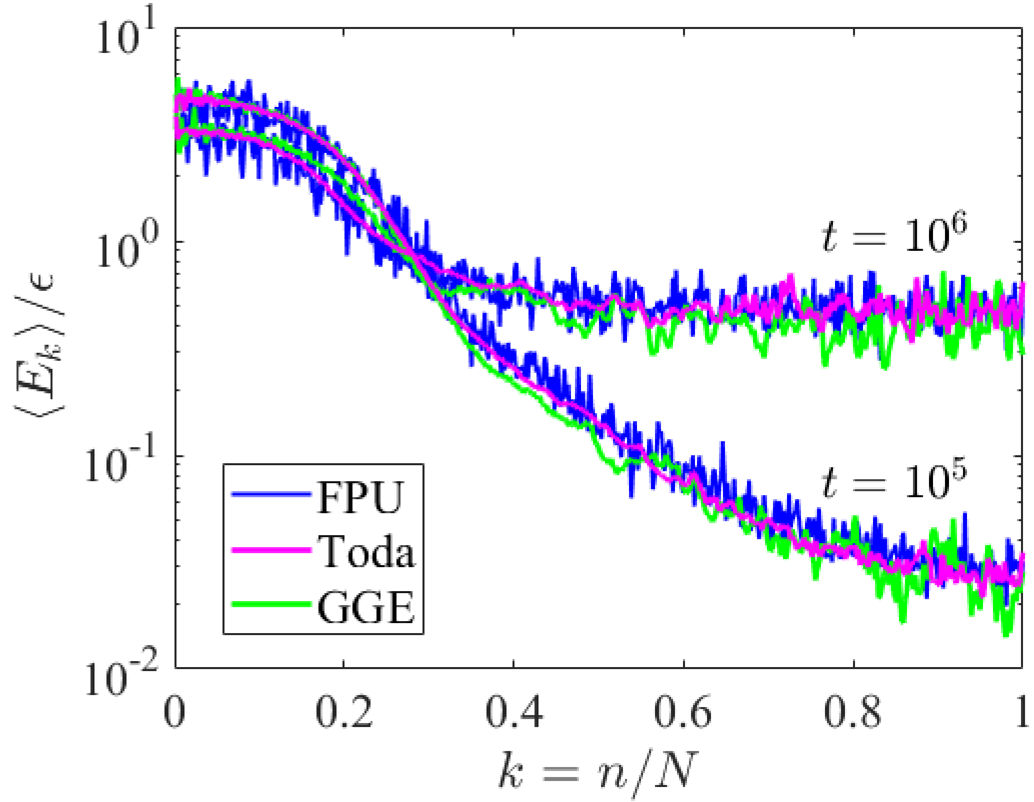
$$\{J_k\}_{k=1}^{N} \to \{\bar{J}_l\}_{l=1}^{10}$$

Gives non-fluctuating quantities and self-averaging





The Slow Dynamics of FPU- generating GGE



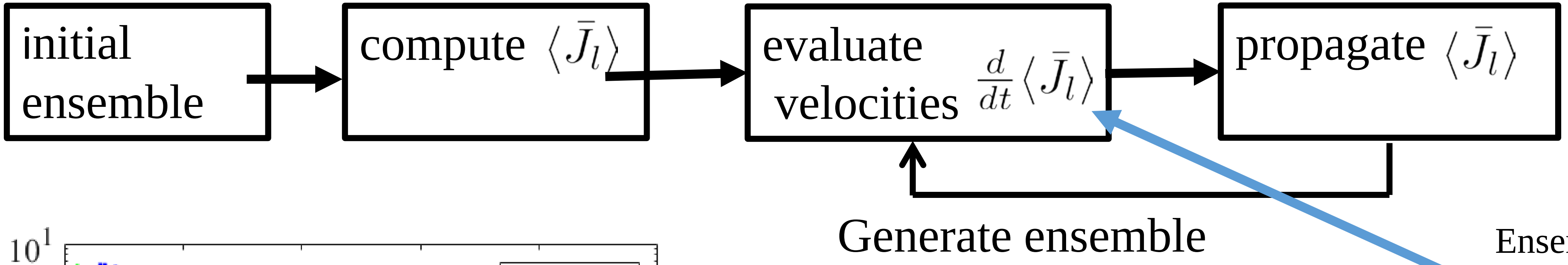
What is the dynamics of $\{\bar{J}_l\}$?
Not easy to compute analytically, however...

Knowing the set $\{\bar{J}_l\}$ we can generate a GGE, and take ensemble average of any observable

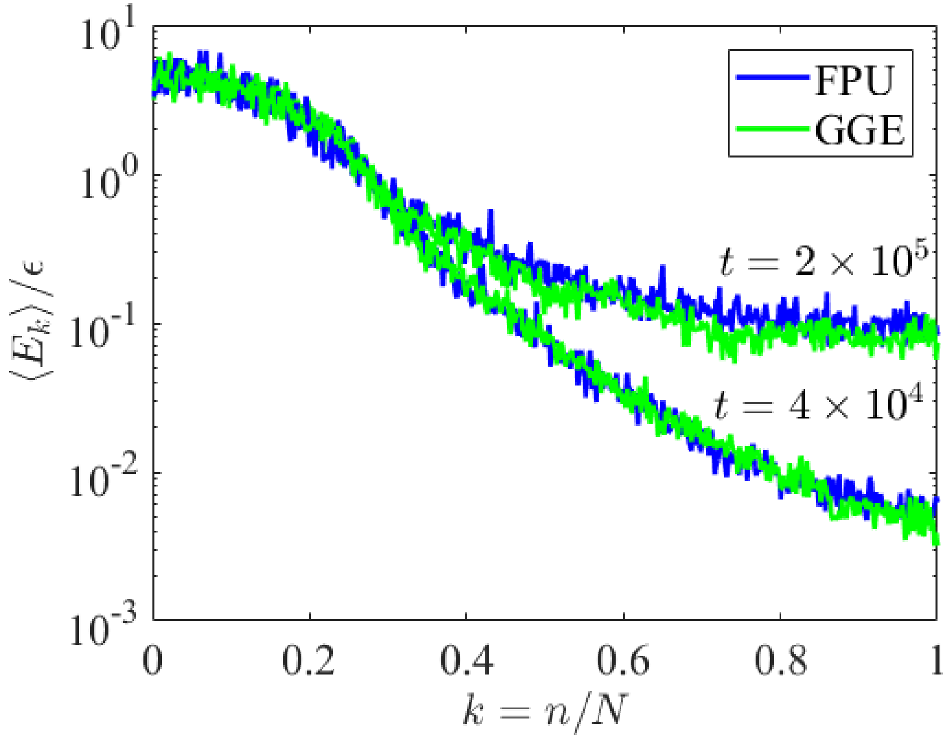
microcanonical GGE: $\delta\left(\bar{J}_l(\vec{x}) - \langle \bar{J}_l \rangle\right)$

The Slow Dynamics of FPU- Fast numerical integration

It should be sufficient to follow the average dynamics of the finite binned values of Toda constants in order to evolve the FPU chain.



Ensemble average/linear fit in a short time window



$N = 511,\ \epsilon = 10^{-3}$

$\{\bar{J}_l\}_{l=1}^{23}$

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Exchange Fluctuation Theorem for quasi-integrable systems

Under the FPU dynamics: the Toda conserved quantities flow slowly due to entropic pressure.

- What is the possibility to *drift to the opposite direction*?

We want to describe, by a fluctuation theorem, the violations of irreversibility as the system thermalizes ($\beta_r \to 0, \text{ except the one related to the energy}$).

- Indeed, we can prove the following (for general quasi-integrable system):

$\to$ The system starts from a GGE state, $\rho(x) \propto e^{-\sum \beta_r J_r(x)}$

$\to$ Let it to evolve by a quasi-integrable dynamics for some t

$\to$ We measure the quantity $u \equiv \sum_r \beta_r \Delta J_r$

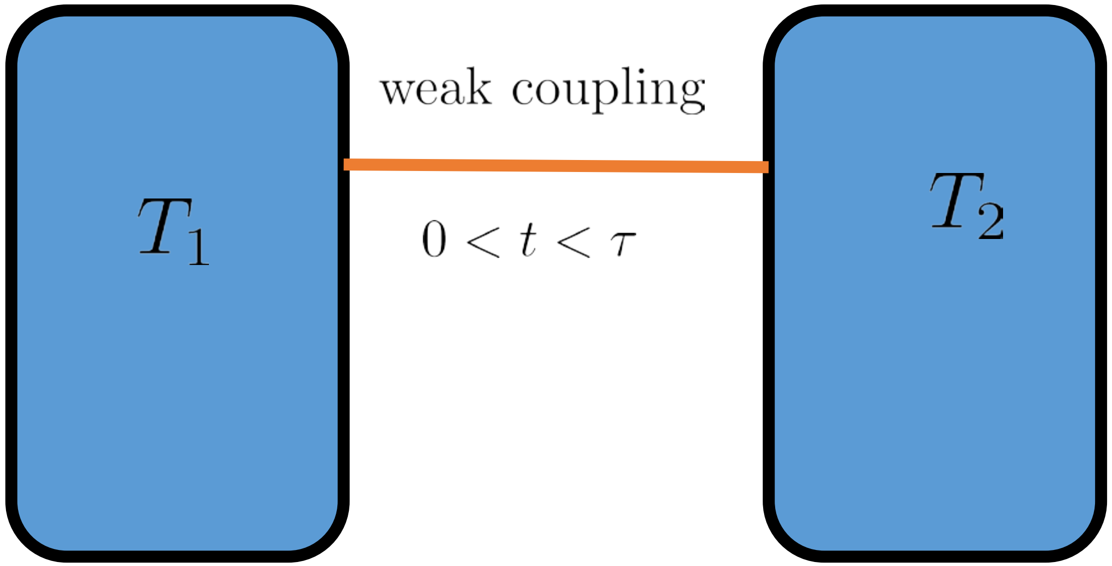
$$\to \quad \boxed{\text{it obeys } \frac{P(u)}{P(-u)} = e^u}$$

Unlike usual FTs, the relation is relevant even for large systems and long times, provided that the breaking of integrability is small.

See also Hickey & Genway (2014); J. Mur-Petit et. al. (2018)

Exchange Fluctuation Theorem for quasi-integrable systems

Simplest GGE setup $\longleftrightarrow$ Jarzynski & Wójcik 2004



initial state: $\rho(x) \propto e^{-\beta_1 E_1 - \beta_2 E_2} = e^{-\beta_+ E_+ - \beta_- E_-}$

$$E_+ = \frac{E_2 + E_1}{2} \qquad \beta_+ = \beta_2 + \beta_1$$

$$E_- = \frac{E_2 - E_1}{2} \qquad \beta_- = \beta_2 - \beta_1$$

$Q \approx \Delta E_2 - \Delta E_1$ heat exchange between the baths

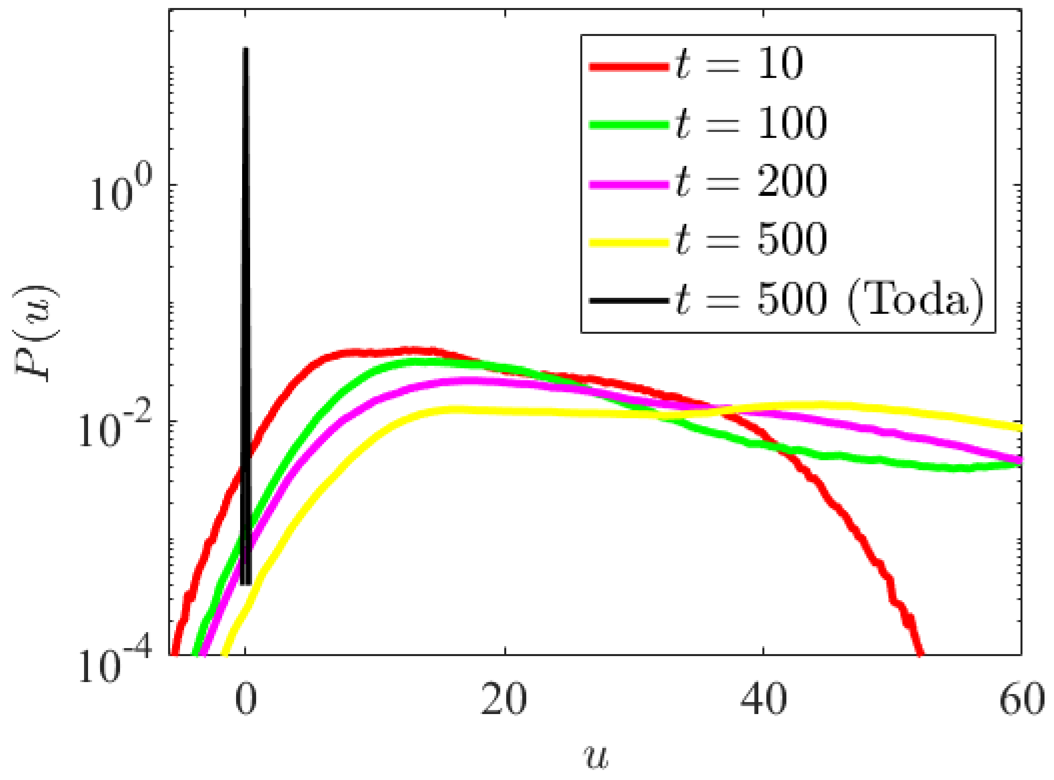
$$\begin{aligned} \Delta E_+ &= 0 \\ \Delta E_- &= Q \end{aligned} \implies \frac{P(\beta_- \Delta E_-)}{P(-\beta_- \Delta E_-)} = e^{\beta_- \Delta E_-} \implies \frac{P(Q)}{P(-Q)} \overset{?}{=} e^{\beta_- Q}$$

This is Jarzynski & Wójcik exchange fluctuation theorem

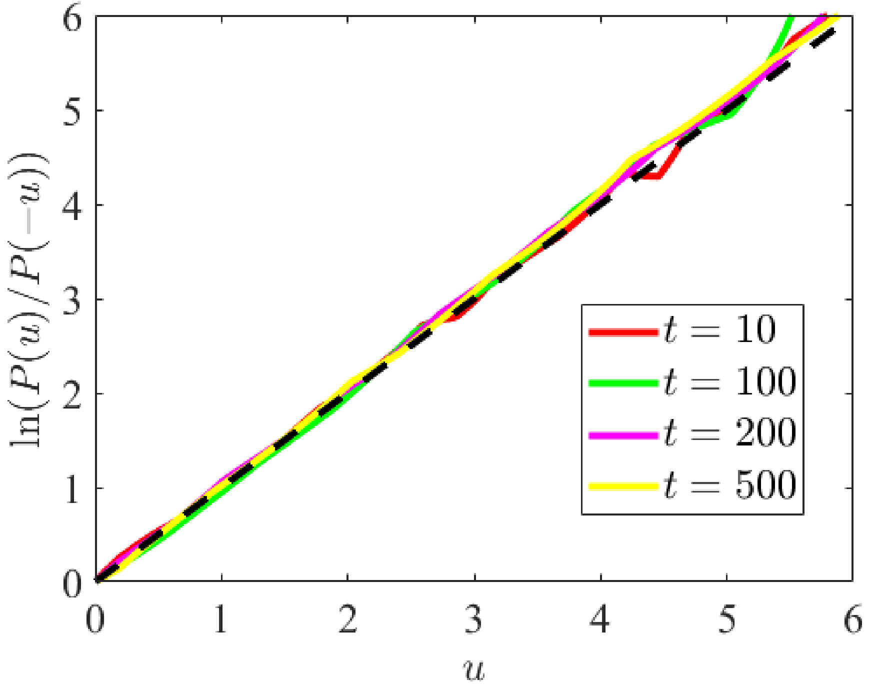
Exchange Fluctuation Theorem for quasi-integrable systems

Example: FPU chain

$$u \equiv \sum_r \beta_r \Delta J_r$$



$N = 15,\ \langle \epsilon \rangle \approx 0.01$



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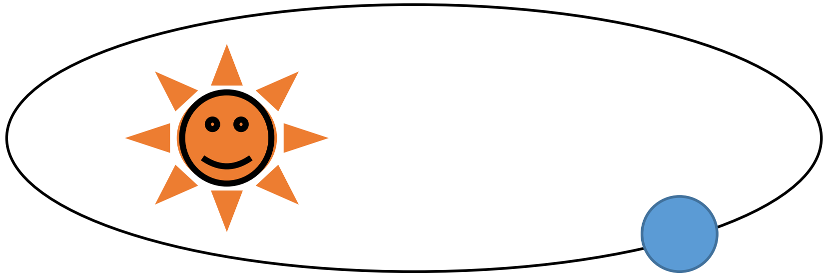
Outlook- Solar System Dynamics

Planetary orbits

almost circular: eccentricity $e \ll 1$

almost planar: inclination $I \ll 1$



Few (8) body dynamics

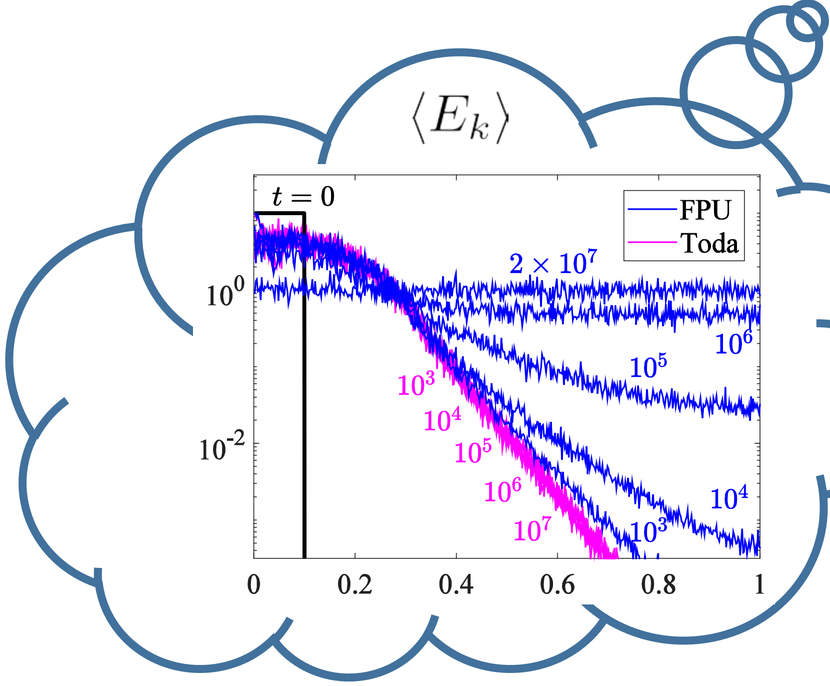
Why statistical mechanics? Chaos

(From numerical simulations)

<u>Two time-scales:</u>

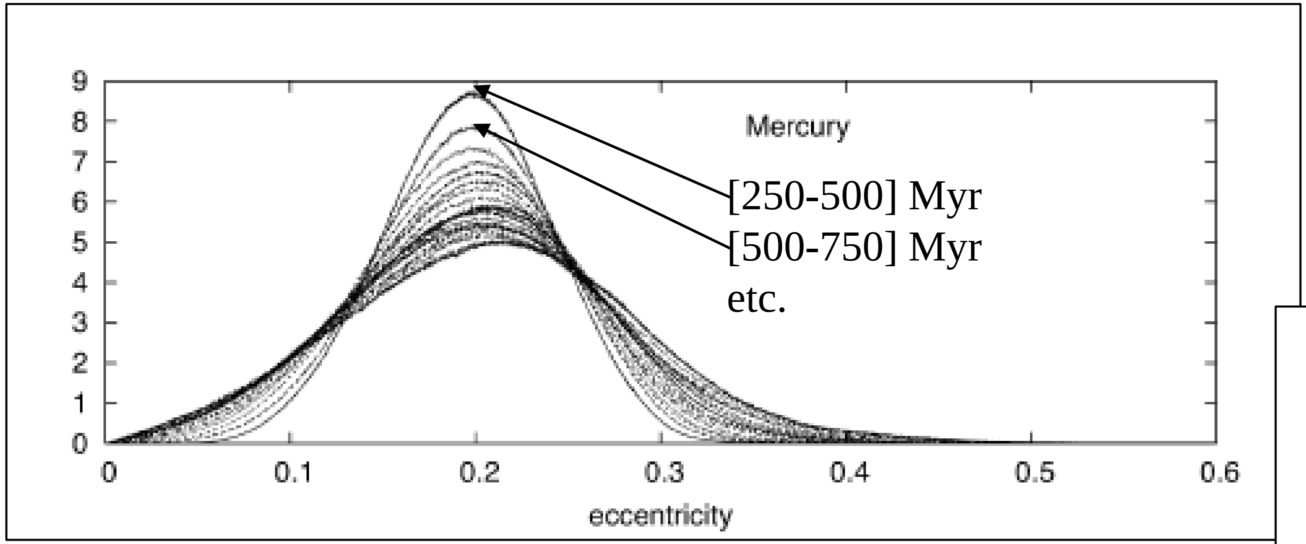
- Fast– chaos (Lyapunov time ~ 5 million years)
- Slow– instability (~ 5 billion years)

Solar System Dynamics



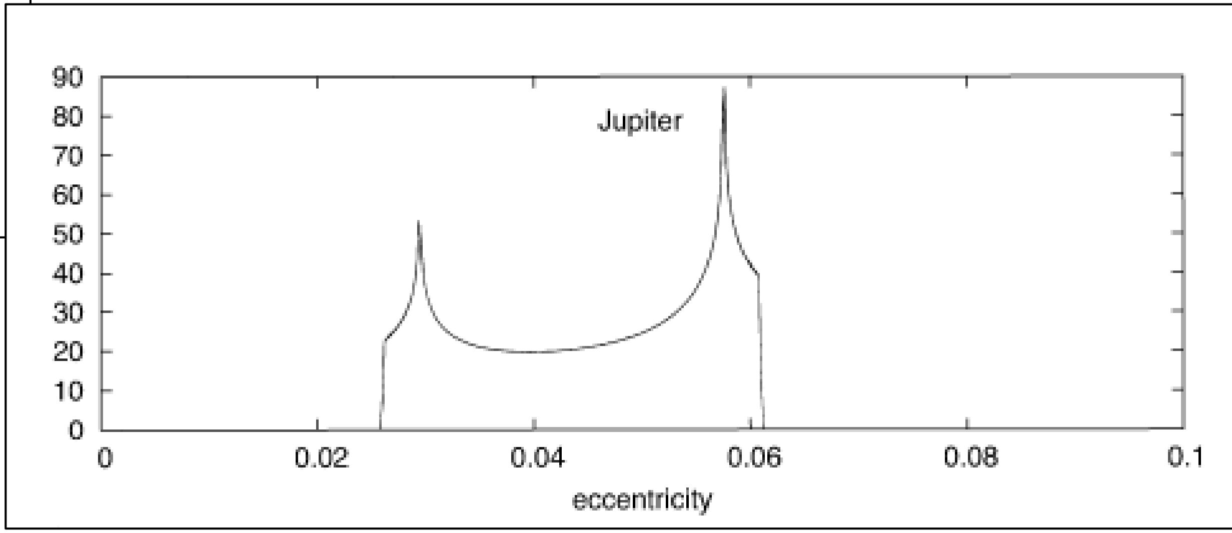
- **Numerical experiment**: Laskar (2008) took 1001 close initial conditions and integrated for 5 Gyrs;

- **Observable:** e.g., eccentricities or inclinations of a planet



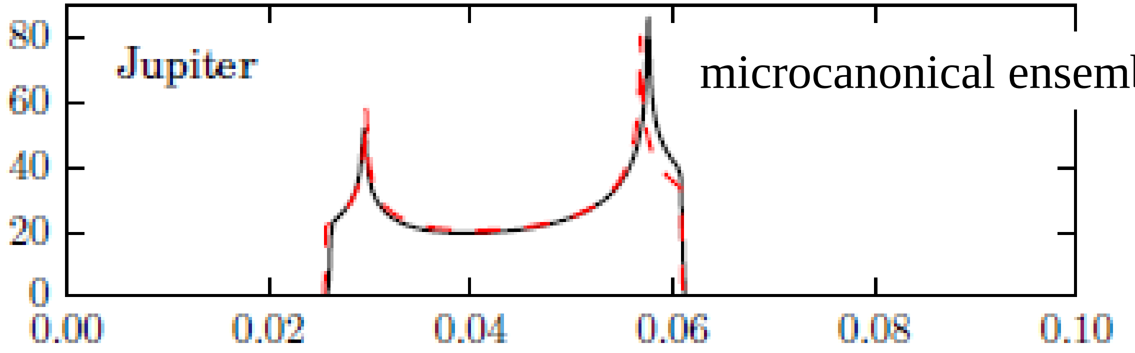
Statistics over consecutive intervals of 250 Myr.

Within 5 Gyr: ~1% of the solutions reach eccentricity of 0.9.

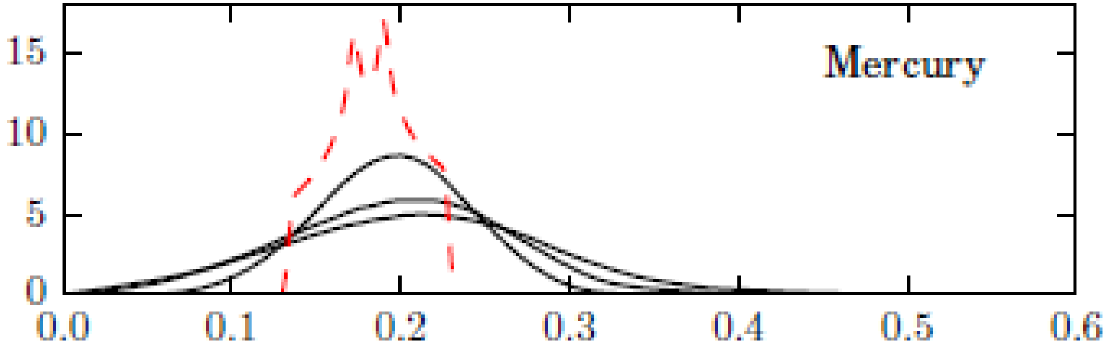


Solar System Dynamics

- **The underlying integrable model:** A good candidate is the Laplace-Lagrange system $(J_\alpha, \theta_\alpha)$



microcanonical ensemble



The outer planets:

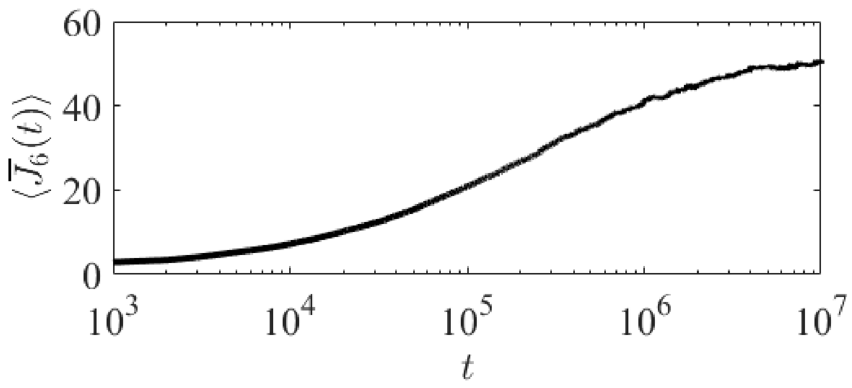
Completely determined by the Laplace-Lagrange tori

Mogavero (2017)

How can we understand this?

Solar System Dynamics

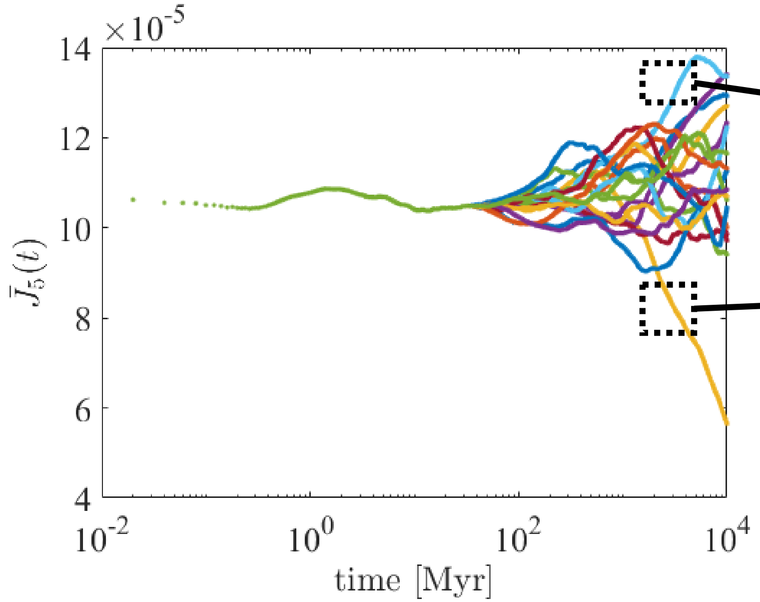
<u>FPU:</u> for large system we have small fluctuations and self-averaging- *drift with small diffusion.*



<u>Solar system:</u> we expect that the noise will be significant- *diffusion, large fluctuations on top of the drift*

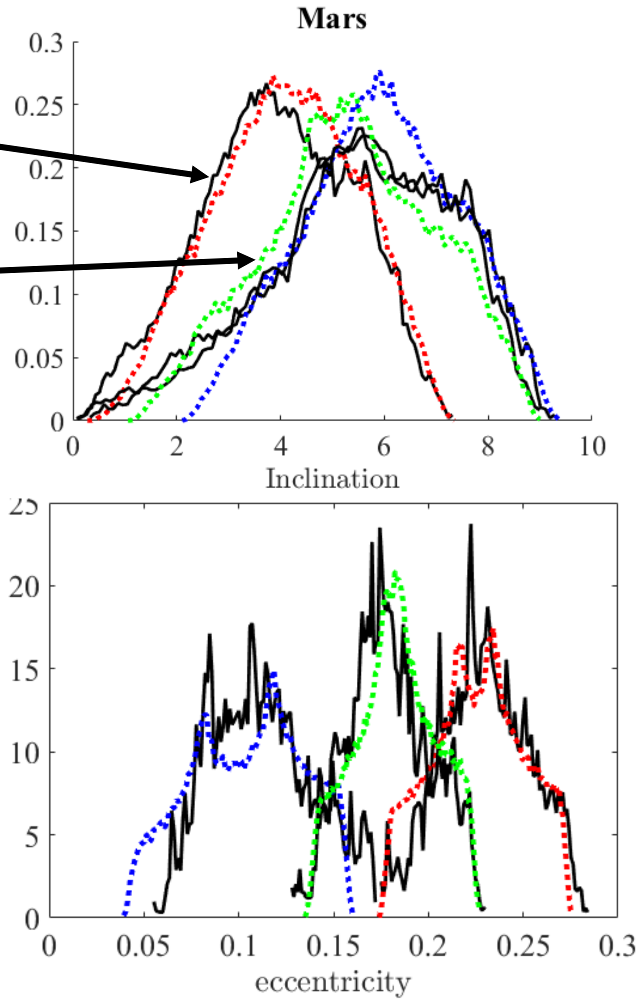
Solar System Dynamics

Solar system: we expect that the noise will be significant- diffusion, large fluctuations on top of the drift



Integrating the secular equations with corrections to Laplace-Lagrange Hamiltonian.

Starting with slightly different initial conditions: diffusion of Laplace-Lagrange action variables



Black: Zoom on the dynamics [2,2.002] Gyrs
Colors: Laplace-Lagrange tori

Summary

- Many-body (deterministic) quasi-integrable systems resemble the features of stochastic ones.

- The slow dynamics of the FPU chain is dictated by the hydrodynamic description of the weakly conserved quantities of Toda.
  - This allows to devise a fast numerical integration.
  - The GGE underlies a fluctuation theorem for quasi-integrable systems.

- Solar system, can we do the same thing?

  A fast integration scheme should also take into account diffusion.

$$\frac{d}{dt}\langle J_l \rangle: \ \mathrm{drift} \qquad \langle J_l(t) J_k(t') \rangle: \ \mathrm{diffusion}$$

Thank you!

Toda (= Thank you in Hebrew)

Fluctuation Theorem → arXiv:1807.08497
Slow Dynamics → In preparation