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Hydrodynamics of Integrable Classical and Quantum Systems

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joint work Benjamin Doyon (Oxford)

1. Integrability

‖ 1D ‖ ‖ translation invariant ‖

EXAMPLE: XXZ chain $\quad \vec{\sigma}_j$ spin at site $j \in \mathbb{Z}$

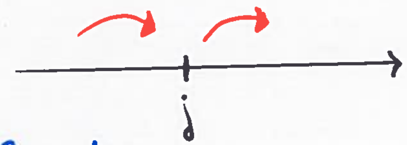
$$H = \sum_j \left(\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta\, \sigma_j^z \sigma_{j+1}^z \right) = \sum_j \underbrace{H_j}$$

energy density local

- conserved charges $Q^{(n)}$: $[H, Q^{(n)}] = 0$, $\quad n = 0, 1, \dots$

and $\quad Q^{(n)} = \sum_j Q_j^{(n)} \quad$ density $Q_j^{(n)}$ has to be quasi-local

$\leadsto \quad i\,[H, Q_j^{(n)}] = J_j^{(n)} - J_{j+1}^{(n)}$ $\quad$ current of $Q^{(n)}$



‖ local conservation law ‖

integrable $\Leftrightarrow$ model has an infinite number of local conservation laws

classical + quantum

abelian and non-abelian charges

- time stationary states $\frac{1}{Z} e^{\vec{\mu} \cdot \vec{Q}}$ $\vec{Q} = \{ Q^{(n)} \}_{n=0,1,\ldots}$

chemical potentials $\vec{\mu} = \{ \mu_n \}_{n=0,1,\ldots}$

Generalized Gibbs Ensemble

2. Hydrodynamics

initial state $\frac{1}{Z} e^{\left[\sum_{n=0}^{\infty} \sum_j \mu_n(\varepsilon j) \, Q_j^{(n)} \right]}$ slow variation $\varepsilon \ll 1$

$$\partial_t \langle Q_j^{(n)}(t) \rangle_{\vec{\mu}} + \nabla \langle J_j^{(n)}(t) \rangle_{\vec{\mu}} = 0$$

$\Downarrow$ in approximation: update $\vec{\mu}(\varepsilon j)$ to $\vec{\mu}(x,t)$ continuum

$$\partial_t \langle Q_0^{(n)} \rangle_{\vec{\mu}(t)} + \partial_x \langle J_0^{(n)} \rangle_{\vec{\mu}(t)} = 0$$

? GGE average $\langle Q_0^{(n)} \rangle_{\vec{\mu}}$? Bethe ansatz

? currents $\langle J_0^{(n)} \rangle_{\vec{\mu}}$? educated guess 2016 | Bertini, Caux, Collura, DeNardis, Doyon, Fagotti, Ilievski, Yoshimura, ...

- complete list of conserved charges is required
- law of large numbers

Examples:

hard rods ⇐

Toda chain $\sum_j e^{-(q_{j+1}-q_j)}$

Calogero-Moser $\sum_{i \neq j} \frac{1}{(q_i - q_j)^2}$

KdV

nonlinear Schrödinger

XY , XXZ

Lieb-Liniger δ-Bose gas $\sum_{i \neq j} \delta(x_i - x_j)$

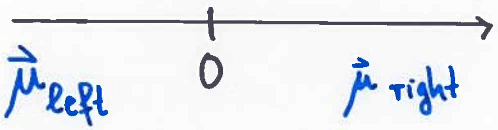
spin 1/2 Hubbard

3. Why hydrodynamics?

- trap potential $\qquad \sum_j V_{ext}(q_j)$

  breaks integrability, **BUT** slow variation

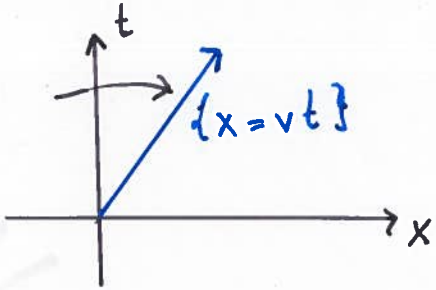
- domain wall initial state



finite number of charges, $\alpha = 1, \ldots, n$, Riemann problem

$\partial_t u_\alpha + \partial_x j_\alpha(\vec{u}) = 0 \qquad$ self-similar $\quad u_\alpha(x,t) = \tilde{u}_\alpha(x/t)$

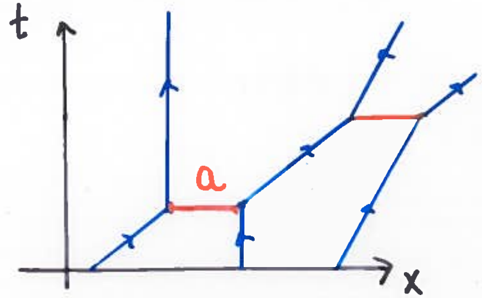
$\tilde{u}_\alpha(v)$



state close to 0, $t \to \infty$, $\tilde{u}_\alpha(0)$

here $n \to \infty$

4. Hard rod fluid



Percus 1969
Dobrushin et al. 1983
Boldrighini, Suhov 1997
domain wall Doyon, H.S. 2017

- velocities are locally conserved integrable

GGE density ρ positions $\{q_j\}$ are correlated

velocity $h(v)dv$ velocities $\{v_j\}$ i.i.d.

$\rightsquigarrow$ $f(x,t;v) \geq 0$

label of conserved field

$$\rho = \int dw\, f(w)$$
$$u = \frac{1}{\rho}\int dw\, w\, f(w)$$

$$\partial_t f(v) + \partial_x \underbrace{v^{\mathrm{eff}}(v)}_{\text{depends on } f} f(v) = 0 \qquad v^{\mathrm{eff}} = v + \frac{a}{1-a\rho}\int dw\,(v-w)\, f(w) = \frac{v - a\rho u}{1-a\rho}$$

- domain wall

$$f(x,0;v) = \begin{cases} \rho_- h_-(v) \,, & x < 0 \\ \rho_+ h_+(v) \,, & x > 0 \end{cases}$$

self-similar

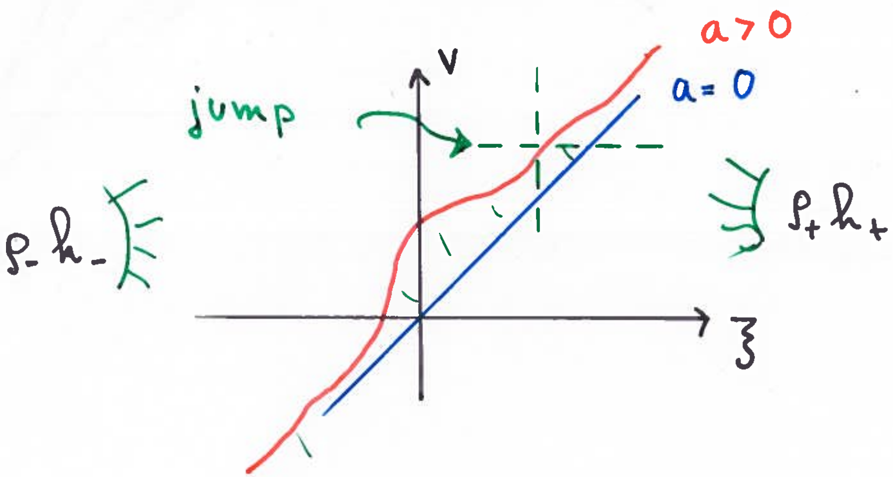
$$f(x,t;v) = \tilde{f}\Big(\underbrace{\tfrac{x}{t}}_{=\,\xi} ; v \Big)$$

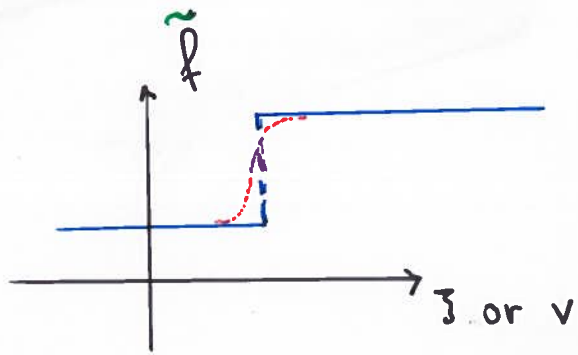
finite n: $\partial_t u_\alpha + \partial_x j_\alpha(\vec{u}) = 0 \rightsquigarrow \partial_t u_\alpha + \underbrace{\sum_{\alpha'=1}^{n} A_{\alpha\alpha'}(\vec{u})}_{\text{determines } \tilde{u}_\alpha} \partial_x u_\beta = 0$

HERE

$$n(v) = \frac{1}{1-a\rho} f(v) \rightsquigarrow \partial_t n + v^{\text{eff}} \partial_x n = 0$$

general





5. Interacting and noninteracting integrable models

noninteracting	interacting
ideal gas	hard rods
harmonic chain	Toda lattice
free lattice bosons / fermions	Lieb-Liniger
XY model with external field (Majorana fermions)	XXZ
Luttinger	Hubbard
⋮	⋮

How are they distinguished?

ONLY ballistic transport	"viscosity" dissipative transport

- current - current correlation   total

$$T_{\alpha\alpha'}(t) = \int dx \, \langle J_\alpha(x,t) J_{\alpha'}(0,0) \rangle^c_{\vec{\mu}}$$

exponential decay in x (Lieb - Robinson)

- Drude weight

$$D_{\alpha\alpha'} = \lim_{t\to\infty} T_{\alpha\alpha'}(t)$$

ensemble dependent

- Onsager matrix

$$L_{\alpha\alpha'} = \int dt \left(T_{\alpha\alpha'}(t) - D_{\alpha'\alpha} \right)$$

little information on decay in t

non interacting $\Leftrightarrow$ $L = 0$

6. Dissipative corrections to hard rods

label velocity v conserved fields $q_v(x,t)$, currents $J_v(x,t)$

GGE density ρ, velocity distribution $h(v)\,dv$, $\Rightarrow f = \rho h$

$$v^{\mathrm{eff}}(v) = \frac{v - a\rho u}{1 - a\rho}$$

dressing operator $(1 - Tn)^{-1}$

$$T\,\psi(v) = -a \int dw\, \psi(w), \quad n\,\psi(v) = \frac{1}{1-a\rho} f(v)\, \psi(v)$$

- static susceptibility

$$\int dx \, \langle q_v(x)\, q_{v'}(0) \rangle_\mu^c = \left((1-nT)^{-1} f (1 - Tn)^{-1} \right)_{vv'}$$

- current - current

$$\Gamma_{vv'}(t) = \delta(t)\, L_{vv'} + D_{vv'}$$

Onsager Drude

$$D_{vv'} = \left((1-nT)^{-1} f \, (v^{\mathrm{eff}})^2 \, (1-Tn)^{-1} \right)_{vv'}$$

$$\langle \phi, L \psi \rangle = \frac{1}{2} \frac{a^2}{1-a\rho} \int dv\, dv' \, f(v) f(v') |v-v'| \left(\phi(v) - \psi(v) \right) \left(\phi(v') - \psi(v') \right)$$

- Navier-Stokes correction $f(x,t;v)$

$$\partial_t f + \partial_x j_{[f]} = \frac{1}{2} \partial_x \left(\frac{a^2}{1-a\rho} \int dw \, |v-w| \left(f(w) \partial_x f(v) - f(v) \partial_x f(w) \right) \right)$$

- entropy production GGE entropy $-\int f \log f \, dv + \rho \log(1-a\rho)$

balance $\partial_t s + \partial_x j_s = \sigma \leftarrow$ production

$$\sigma_{[f]} = \frac{1}{4} a^2 \frac{1}{1-a\rho} \int dv\, dw \, |v-w| f(v) f(w) \left(\frac{1}{f} \partial_x f(v) - \frac{1}{f} \partial_x f(w) \right)^2$$

De Nardis, Bernard, Doyon June 2018 extend to

$\|$ large class of integrable models $\|$

not only Drude weight but also Onsager matrix

have a common structure

- classical, Bose, Fermi
- scattering amplitude
- list of conserved charges

7. Effective hard rod dynamics

dressing transformation $\quad (1 - Tn)^{-1}$

$$T\psi(v) = \int \varphi(v-w)\,\psi(w)$$

scattering amplitude

Example: Lieb-Liniger

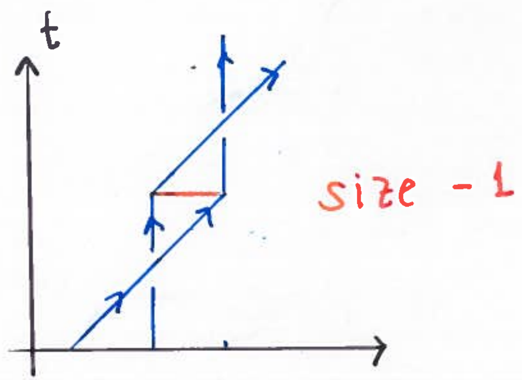
$$\varphi(v) = \frac{4c}{v^2 + 4c^2} \qquad c > 0$$

strength of δ

(hard rods $\varphi_{hr}(v) = -a$)

SUGGESTS

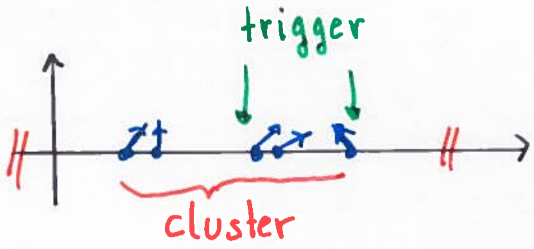
hard rods with
v dependent size



// classical simulation of the hydrodynamics of a quantum system //

(faster than solving coupled PDEs)

A) how to define?



run full scattering
incoming → outgoing

instantaneous

B) time-stationary states?

how is the quantum susceptibility achieved?

matrix C

Summary

- locally conserved fields (densities)
- Euler type hydrodynamics $\langle \mathfrak{z}_\alpha \rangle_{\vec{\mu}}$
- domain wall / trap
- viscosity / Onsager matrix $\Leftarrow$ hidden by Euler
- effective hard rods