Perturbed CFT: Integrability, non-integrability and the level spacing conjecture

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Universal behaviour of the critical point in second order phase transition are described by CFT.

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Universal behaviour of the critical point in second order phase transition are described by CFT.

What happens if we move away from the critical point?

\[ A = A_{\text{CFT}} + g_1 \int d^2x \phi_1(x) + g_2 \int d^2x \phi_2(x) \]

For Ising model for example

- \( A_{\text{CFT}} \) describes the critical point (\( T = T_c \) and \( B = 0 \))
- \( g_1 \sim \frac{T - T_c}{T_c} \)
- \( g_2 \sim B \)
2D CFT has an infinite number of conserved quantities $\rightarrow$ integrability

What about the perturbed case? It turns out that for some specific single deformations an infinite number of conserved quantities survive

For a generic single perturbation or multiple perturbation the theory is usually non integrable
How do we threat the integrable case?
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Reinterpret the theory as a scattering theory in 1+1 dimensions

- Integrability restricts the dynamics such that only elastic scattering are allowed
- Moreover multiparticle scattering processes are factorized in two particles processes

Matching the conserved quantities from CFT consideration with the ones of the scattering the theory determines the S-matrix

For non-integrable perturbation, such construction is not possible
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Reinterpret the theory as a scattering theory in 1+1 dimensions
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  - Moreover multiparticle scattering processes are factorized in two particles processes

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For non-integrable perturbation, such construction is not possible
  - Is there general tool to study perturbed CFT?
Truncated space approach

- Consider the Hamiltonian $H = H_0 + H_{\text{pert}}$
- Eigenstates and eigenvalues of $H_0$ are known
- Truncate the spectrum at a certain number of levels
- Evaluate the matrix element of $H$ on the truncated base and diagonalize
\[ [L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12}n(n^2 - 1)\delta_{n+m,0} \]

Highest weight vectors
\[
L_0|\Delta\rangle = \Delta|\Delta\rangle \\
L_n|\Delta\rangle = 0 \quad n > 0
\]

Descendant states
\[
|\psi\rangle = L_{-n}^{k_n} \cdots L_{-1}^{k_1} |\Delta\rangle \\
L_0|\psi\rangle = \Delta + \sum_{n} (k_n n)|\psi\rangle
\]

In terms of fields
\[
|\Delta\rangle = \lim_{z \to 0} \phi_\Delta(z)|0\rangle
\]
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In terms of fields

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**Minimal CFT**

- Defined by two coprime integers \( p, q \) \( \rightarrow \) central charge \( c \) and field dimensions
- Finite number of primary fields
Consider the Hamiltonian formulation of the perturbed model, on a cylindrical geometry (for $\Delta = \bar{\Delta}$)

$$H = \frac{2\pi}{R} \left( L_0 + \bar{L}_0 - \frac{c}{12} \right) + g \int_0^R dx \phi_{\Delta}(x)$$

The matrix elements on the conformal basis

$$\langle \psi_i \vert H \vert \psi_j \rangle = \delta_{i,j} E_i + gR \delta_{p_i,p_j} \langle \psi_i \vert \phi_{\Delta}(0) \vert \psi_j \rangle$$

where

$$E_i = \frac{2\pi}{R} \left( \Delta_\psi + \bar{\Delta}_\psi - \frac{c}{12} \right)$$

$$P_i = \frac{2\pi}{R} \left( \Delta_\psi - \bar{\Delta}_\psi \right)$$
Matrix elements of the perturbation are related to three-point correlation functions

\[ \langle \psi_i | \phi_\Delta(0) | \psi_j \rangle \rightarrow \langle 0 | \psi_i(z_1, \bar{z}_1) \phi_\Delta(z_2, \bar{z}_2) | \psi_j(z_3, \bar{z}_3) | 0 \rangle \]

Correlation function involving descendant fields are proportional to correlation function between primary fields.

Finally, correlation function involving primary fields are determined by the structure constants of the OPE.

We have then

\[ \langle \psi_i | \phi_\Delta(0) | \psi_j \rangle \propto \left( \frac{2\pi}{R} \right)^{2\Delta} C_{\Delta_a, \Delta, \Delta_b} \]

The proportionality constant is completely determined by the Virasoro algebra and the conformal dimensions of the fields.
- TCSA on every possible unitary minimal model
- Arbitrary number of perturbations
- Calculation of structure constants and character expansion
- Decomposition of the matrix elements in independent subspaces
- Optional NRG on the levels
TCSA on every possible unitary minimal model
Arbitrary number of perturbations
Calculation of structure constants and character expansion
Decomposition of the matrix elements in independent subspaces
optional NRG on the levels

For a perturbation $\phi_\Delta$ matrix elements of the perturbation are related to $C_{\Delta_1,\Delta,\Delta_2}$
The matrix indicates independent subspaces.

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Matrix $C_{\Delta_1,\epsilon,\Delta_2}$ for a $H_{M_3} + g \int dx \epsilon(x)$ (Ising model at $t \neq t_c$)
Tricritical Ising model at $t \neq t_c$

\[
(H_{M4} + g_1 \int dx \phi_{1/10})
\]

Tricritical Ising model at $t \neq t_c$
and non-zero magnetic field

\[
(H_{M4} + g_1 \int dx \phi_{1/10} + g_2 \int dx \phi_{3/80})
\]
Predictions on the number and structure of vacua
For example for

\[ M_7 + g_1 \phi_{1,3} + g_2 \phi_{1,2} \]

\[ g_1 < 0 \quad g_2 = 0 \]
\[ g_1 < 0 \quad g_2 > 0 \]
\[ g_1 < 0 \quad g_2 < 0 \]
Other quantities can be evaluated

- Form factors $\rightarrow \langle 0 | \phi(0) | \theta_1 \ldots \theta_n \rangle$
- S-Matrix

\[ m_i R \sinh(\theta_i) + \sum_{j \neq i} \delta(\theta_j - \theta_i) = 2\pi l_i \]

\[ E(R) = \sum_{i=1}^{N} m_i \cosh(\theta_i) \]

\[ \delta(\theta_i - \theta_j) = -i \ln S(\theta_i - \theta_j) \]
Level spacing statistics conjecture

- A integrable quantum system has a Poissonian level spacing distribution
- A non-integrable system has GOE level spacing distribution

The level spacing distribution is evaluated as follow:

- From the level energy levels build the Integrate DOS
  \[ N(E) = \sum_i \theta(E - E_i) \]
- Interpolate the integrated DOS skipping a certain number of point, obtaining \( \bar{N}(E) \)
- Calculate \( s_i = \bar{N}(E_i) - \bar{N}(E_{i-1}) \)
  \[ \bar{N}(E_i) - \bar{N}(E_{i-1}) = \int_{E_{i-1}}^{E_i} dE \rho(E) \approx (E_i - E_{i-1})\rho(E_{i-1}) \]
- Build the Histogram of \( s \)
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Level spacing statistics

\[ H_{M7} + \phi_{1,2} \]

\[ H_{M7} + \phi_{1,2} + \phi_{2,2} \]
$$H_{M_4} + \int dx \epsilon(x)$$

R=0.2

R=0.8

R=3.2