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SCUOLA INTERNAZIONALE SUPERIORE DI STUDI AVANZATI - INTERNATIONAL SCHOOL FOR ADVANCED STUDIES
I-34014 Trieste ITALY - Via Beirut 4 - Tel. [+]39-40-37871 - Telex:460269 SISSA I - Fax: [+]39-40-3787528

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This is a collection of problems given in recent entrance examinations in various SISSA Sectors, which may be relevant for the Statistical Physics Curriculum.

Problem B. Classical solutions in Field Theory

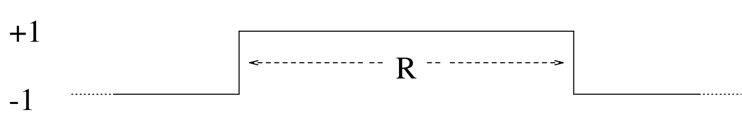
Field Theories may have degenerate vacua and kinks, i.e. interpolating field configurations among them. Let

$$H = \int dx \left[\frac{1}{2} \left(\frac{\partial \varphi}{\partial x} \right)^2 + U(\varphi) \right] \tag{1}$$

be the classical hamiltonian for the static (time–independent) configurations of φ, a scalar field in one space dimension.

1. Take $U(\varphi) = \frac{g}{4}(\varphi^2 - 1)^2$ $(g > 0)$. Derive the differential equation satisfied by the static configurations and prove that $\phi(x) = \pm \tanh[\sqrt{\frac{g}{2}}(x - x_0)]$ is a solution ($\phi_+(x)$ being a kink and $\phi_-(x)$ being an anti–kink). Compute its energy density, interpret physically the profile of this curve and determine how the total energy of the kink depends on g.

2. For the above model, consider the configuration made of a kink and an anti–kink separated by a large distance R (the precise mathematical expression of this configuration is inessential and can be taken as a step function of jumps ± 1, see figure below). Let us switch an external source h on, so that $U \to U + \delta U$, with $\delta U = h\varphi$. Evaluate the effect of the new interaction on the (unperturbed) configuration drawn in the Figure by estimating the behaviour of its energy for large R and determine whether an attractive or repulsive potential between the kink and the antikink will be generated.



Problem 1

Consider a gas of independent particles. Each particle can be in a state $k = 0, 1, 2, \ldots$. In a time interval dt, for each particle one of the following processes may occur:

- with probability ηdt the state *spontaneously* increases by one: $k \to k + 1$

- with probability $k dt$ the state *spontaneously* decreases by one: $k \to k - 1$

• Write the master equation for this process and find the stationary state distribution $P_0(k)$ of k. (*Hint:* use the generating function $P(z) = \sum_{k=0}^{\infty} P(k)\, z^k$.) Now consider a case where we switch on the interaction between particles by introducing the following additional process:

- in a time interval dt with probability ξdt particle i collides with a particle j chosen at random, and if $k_j > 1$ then $k_i \to k_i + 1$, otherwise k_i is unchanged. The interaction leaves k_j unchanged.

Consider the mean field approximation where the probability $P_\xi(k, t)$ has the same functional form of $P_0(k)$ but the average $\langle k \rangle = c(t)$ depends on time.

• Discuss qualitatively the behavior of the system, within this approximation, as a function of the interaction parameter ξ for $\eta \ll 1$.

• Will the gas always reach the same stationary state independently of the initial condition?

Problem 2

Consider a system with two energy levels, $E_A = 0$, $E_B = \epsilon > 0$ occupied by N particelle that are in thermal equilibrium at a temperature $T = 1/\beta$ ($K_B = 1$).

1. Calculate the probability, P_n to find n particles in the ground state (state A). Assume that the particles are classical (i.e. subject to Boltzmann statistics). Then calculate the average energy for the system , $\langle E \rangle$, specific heat and average number of particles in the ground state $\langle n \rangle$ as a function of β. Summarise your results in a qualitative sketch for the behaviour of the calculated quantities.

2. Repeat the above calculation for particles obeying Bose statistics. Compare the behaviour of the calculated quantities for $T \to 0$ and $T \to \infty$ for the two cases of Boltzmann and Bose statistics.

3. Assume that detailed balance is satisfied, and that $W(n \to n')$ indicates the rate of transition from n to n' particles in the ground state. Calculate

$$\frac{W(n \to n-1)}{W(n-1 \to n)} \tag{1}$$

   in the classical and quantum (Bose) case and discuss the differences.

4. Assume that

$$W(n \to n-1) = \frac{1}{W(n-1 \to n)} \ . \tag{2}$$

   and that all transition rates $W(n \to n')$ are zero if $|n - n'| > 1$. At time $t = 0$ the system is prepared with all particles in the excited state (state B). Calculate how P_n increases at early times ($t \approx 0$) in the classical and quantum case.

Problem C. Extimate of Ground State Energy in Quantum Mechanics

Exact results in Physics are quite rare but a quite good estimate of physical quantities can be often obtained by means of some approximation schemes. The aim of this exercise is to apply two of these approximations to a simple example of a quantum mechanical system. In the following h is the usual Planck constant and $\hbar = \frac{h}{2\pi}$.

Consider the motion of a quantum particle of mass m on a real axis $-\infty < x < +\infty$ with Hamiltonian given by

$$H = \frac{p^2}{2m} + k \mid x \mid \quad , \quad k > 0$$

1. Discuss the symmetry of the Hamiltonian, the nature of the spectrum (if continuous or discrete) and its degeneracy. Using dimensional analysis, construct a quantity $\mathcal{E}$ in terms of $\hbar$, m and k which has the dimension of an energy and which can be used as the scale for all the energy eigenvalues.

2. Using the uncertainty principle $\Delta p \, \Delta x \geq \hbar$, give an estimate of the ground state energy E_0 of the system and the size of the corresponding wave function.

3. Using the semi-classical quantization condition

$$\oint p(x) \, dx = 2\pi \left(n + \frac{1}{2} \right) \hbar$$

find the (approximate) energy levels E_n of the system. In particular, calculate E_0 and compare with the result of point (3).

1 Spectrum of an half-harmonic oscillator

Consider an harmonic oscillator in one dimension:

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2}x^2$$

and its eigenvalues, ε_n, and eigenfunctions, $\phi_n(x)$.

Suppose that a rigid wall [$v(x) = +\infty$ for $x \leq 0$, $v(x) = 0$ for $x > 0$] is then inserted at the origin, and consider now eigenvalues and eigenfunctions of the resulting asymmetric potential well.

i) What is the energy spectrum of this half-harmonic oscillator? Write the eigenfunction of its M-th excited state in terms of $\phi_n(x)$.

ii) Consider the expansion of this state in terms of eigenvectors of the original oscillator, $\phi_M^{half}(x) = \sum_{n=0}^{\infty} c_n \phi_n(x)$. Give the explicit expression for the expansion coefficients, c_n, on odd-numbered excited states of the original oscillator.

iii) From the expression for the energy of the eigenstate find a sum-rule that must be satisfied by even-numbered coefficients.

Suppose that the system is prepared in the ground state of the half harmonic oscillator and that at a given moment the delta-function potential is removed instantaneously.

iv) Is the probability for the system to be found at a later time on the ground state of the complete oscillator more or less than 25 % ?

3 Statistical Mechanics problem

A classical system can exist in N states $i = 1, ..., N$ with energies $E_1, ..., E_N$. Let $K_{i,j}$ be the transition rate from state j to state i and $P_i(t)$ the probability to be in state i at time t. If the time evolution is written in the form $dP/dt = -HP$

1. determine H in terms of K's.
2. Is probability conservation preserved by time evolution?

If $K_{i,j} = K_{j,i}$

3. show that the eigenvalues of H can not be negative.
4. Is there always a zero eigenvalue and when is it unique?

If the system is in contact with a thermal bath at temperature T and detail balance is satisfied

5. determine the similarity transformation which makes H symmetric.
6. Under which conditions $P_i(t)$ reaches its equilibrium value at large times?
7. Calculate the time evolution for a 2-state system with arbitrary initial conditions and the time dependence of the average energy.

4 A two–component quantum mechanics problem

Consider the one–dimensional quantum problem with an Hamiltonian

$$H = \frac{1}{2}\left(p^2 + W^2(x)\right)\mathbf{1} + \frac{\hbar}{2}\frac{dW}{dx}\sigma_{\mathbf{3}} \ , \tag{9}$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are Pauli matrices which satisfy $[\sigma_l, \sigma_m] = 2i\,\epsilon_{lmn}\,\sigma_n$. As usual, $p = -i\hbar\frac{d}{dx}$ with commutation relation $[f(x), p] = i\hbar\dfrac{df(x)}{dx}$. We assume that $|W(x)| \to \infty$ as $|x| \to \infty$.

1. Prove that the hermitian operators

   $$Q_1 = \frac{1}{2}\left(p\,\sigma_1 + W(x)\,\sigma_2\right) \quad ; \quad Q_2 = \frac{1}{2}\left(p\,\sigma_2 - W(x)\,\sigma_1\right) = -i\sigma_3 Q_1$$

   express a symmetry of the system, i.e. they commute with the Hamiltonian

   $$[Q_i, H] = 0$$

   Hint. It may be useful the following identity for the commutator:

   $$[AB, CD] = AC[B, D] + A[B, C]D + C[A, D]B + [A, C]DB$$

2. Prove that the Hamiltonian can be expressed as $H = \frac{1}{2}Q_1^2$ or, equivalently, as $H = \frac{1}{2}Q_2^2$.

3. Show that the eigenvalues E_n of the Hamiltonian (10) satisfy the condition $E_n \geq 0$ and that the exact ground state wave function (with $E = 0$) can be expressed as

   $$\psi(x) = \exp\left(\int_0^x dy \frac{W(y)}{\hbar}\sigma_3\right)\psi(0)$$

   provided $\psi(x)$ is normalizable. Discuss the conditions that $W(x)$ and $\psi(0)$ have to satisfy in order that this happens.

4. Prove that for $W(x)$ given by $= W(x) = gx^{2n}$ $(n = 1, 2, \ldots)$ the ground state energy E_0 is instead strictly positive and that the symmetry generated by Q_1 and Q_2 is spontaneously broken.

Problem 1

A classical harmonic oscillator of mass m and spring constant k is known to have a total energy E, but its starting time is completely unknown.

1. Find the probability distribution density $p(x)$, where $p(x)dx$ is the probability to find the mass in the interval $[x, x + dx)$.

2. Calculate $p(x)$ in the ground state of the quantum case.

3. Compare and discuss cases 1 and 2.

Problem 2

Consider a chain of N particles with spin 1/2 whose Hamiltonian is

$$\mathcal{H} = -K \sum_{i=1}^{N} \sigma_i^z \, \sigma_{i+1}^z - \Gamma \sum_{i=1}^{N} \sigma_i^x \tag{1}$$

where σ_i^α ($\alpha= x$, y, z) are the Pauli matrices and K and Γ are non-negative constants.

1. For the case $\Gamma = 0$ calculate the free energy per site, f,

$$f = -\frac{1}{N\,\beta} \ln \mathrm{Tr}\; e^{-\beta\,\mathcal{H}} \tag{2}$$

   the average energy and the specific heat.

2. For the case $\Gamma \neq 0$ estimate the ground state energy using a trial wave-function

$$|0\rangle = \otimes_i \; |m\rangle_i \tag{3}$$

   where

$$|m\rangle_i = \sqrt{\frac{1+m}{2}}\; |+\rangle_i + \sqrt{\frac{1-m}{2}}\; |-\rangle_i \qquad\qquad |m| \leq 1 \; , \tag{4}$$

   and $|\pm\rangle_i$ are the eigenstates of σ_i^z.

3. What is the physical meaning of the parameter m ? What is the optimal choice of the parameter m and how does it depend on K and Γ?

4. Compare the two cases, 1 and 2.

Definition of Pauli matrices:

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{5}$$

Problem C. Statistical Mechanics

Let us consider a quantum system of volume V described by the Hamiltonian H. Let $e_m(V)$ $(m = 0, 1, \ldots, \infty)$ be the energy spectrum. The system is put in contact with a heat reservoir at temperature T.

1. Prove that the expectation value of H, denoted by $E(T, V)$, is a monotonic function of the temperature T.

2. Assume that the energy levels are homogeneous functions of the volume V according to the formula

$$e_m(\lambda V) = \lambda^x e_m(V) \ .$$

Show that $E(T, V)$ satisfies a differential equation of the form

$$\left(a\,T \frac{\partial}{\partial T} + b\,V \frac{\partial}{\partial V} + c \right) E(T, V) = 0 \ .$$

Determine a, b and c.

Problem 1

Consider a one-dimensional chain consisting of $n >> 1$ segments as illustrated schematically in the figure.

Let the length of each segment be a when the long dimension of the segment is parallel to the chain and zero when the segment is vertical (i.e. long dimension normal to the chain direction). Each segment has just two states, a horizontal orientation and a vertical orientation, and each of these states is not degenerate, The distance between the chain ends is nx.

1. Find the entropy of the chain as a function of x.
2. Obtain a relation between the temperature T of the chain and the tension F which is necessary to apply to mantain the distance nx, assuming the joints turn freely.
3. Under which conditions does your answer lead to Hook's law?

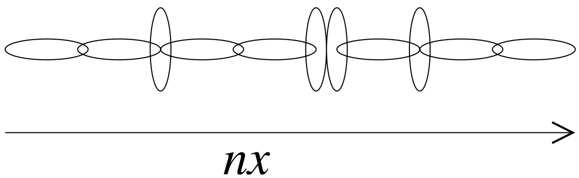


Figure 1: Schematic drawing of a possible chain configuration

Problem 2

Consider a non-relativistic gas of identical (classical) particles in a box. Let $f(\vec{v})\, d^3v$ be the mean number of particles per unit volume with velocity whose components are in the range $(v_\alpha, v_\alpha + dv_\alpha)$ with $\alpha = x, y, z$.

1. determine the number of particles with velocity in the range $(v_\alpha, v_\alpha + dv_\alpha)$ with $\alpha = x, y, z$, striking a unit area in the box wall per unit time;
   assume equilibrium conditions.

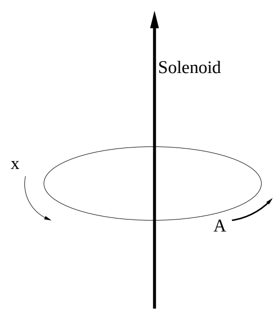
2. explain why $f(\vec{v}) = \hat{f}(v)$ (where $v =| \vec{v} |$). Calculate the total number of particles striking the unit area in the box walls, J_T (= flow). Express your result in terms of the mean number of particle per unit volume, n, and mean particle velocity, $< v >$;

3. express the pressure, P, in terms of n, the mass of the particles, m, and $< v^2 >$;

4. if f is the Maxwellian distribution calculate P (point 3) in terms of n and the temperature, T, and J_T (point 2) in terms of P, m and T.

Problem 2. A ring with magnetic flux

Consider a one-dimensional ring of length L with coordinate x ($0 \leq x \leq L$), and a magnetic field confined in a solenoid through its center, as shown in figure. The vector potential A can be considered constant on the ring and the magnetic flux inside the ring is $\Phi = LA$. For a single particle, setting



$\hbar = e = c = 1$, the Hamiltonian is simply:

$$H = \frac{1}{2m}(-i\partial_x + A)^2$$

and one should solve $H\psi(x) = E\psi(x)$ with the requirement that $\psi(x+L) = \psi(x)$.

1. Show that the field A can be eliminated by considering the wavefunction:
$$\psi(x) = e^{-iAx}\Psi(x)$$

2. What boundary conditions must be satisfied by the wavefunction $\Psi(x)$? Determine accordingly the spectrum of H for a generic flux Φ.

3. Which values of the flux Φ are consistent with a ground state with no current flowing? [Recall that the current is $j(x) = -(1/2m)\psi^*(x)(-i\partial_x + A)\psi(x) + \text{c.c.}$]

4. Consider now *two non-interacting electrons* with opposite spins subject to the same H. For which values of the flux Φ do you have a ground state with no current flowing?

Problem 1

Consider a collection of 2 large spheres and n "small" spheres. The centres of the spheres can stay only on the vertices of a regular polygon of N sides, each of length a. The diameter of each of the two (identical) large spheres is $1.8a$ while that of the (identical) small spheres if $a/2$. The spheres are impenetrable therefore, due to excluded volume effects: (i) no two spheres can occupy the same site (vertex); (ii) two large spheres cannot occupy neighbouring sites; (iii) no small sphere can occupy a site adjacent to a large one.

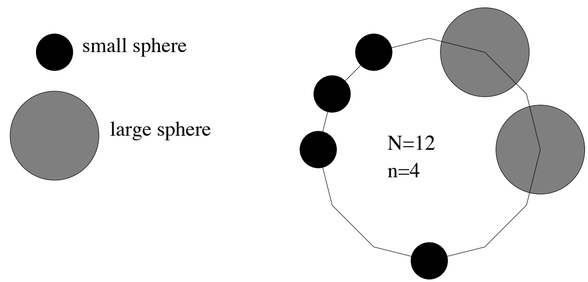


Figure 1: Example of allowed configuration of two large spheres and $n = 4$ small spheres on a ring of $N = 12$ vertices.

Assume that the number of sites, N, is even and that $N >> n >> 1$.

1. Calculate the number of microstates of the system when the centres of the large spheres are at separation $2a$, $3a$, $4a$, ... $N/2a$. (The separation is the shortest distance along the ring going either clockwise or counter-clockwise).

2. Assuming the equiprobability of the microstates, calculate the probability that the large spheres are at separation $2a$, $3a$, $4a$... . What is their most probable separation? What is the least probable separation?

3. Discuss how the average separation of the large spheres depends on n at fixed N. How would it be affected by temperature in the canonical ensemble? Explain and comment your results.

4. Calculate the partition function of the system, $\mathcal{Z}$, and estimate $\partial \ln \mathcal{Z}/\partial n$ and $\partial \ln \mathcal{Z}/\partial N$. Discuss the physical significance of your results.

B.3. Three-Dimensional Harmonic Oscillator

Given a tridimensional harmonic oscillator, whose Hamiltonian is written as

$$\mathcal{H} = \frac{1}{2}\omega\left[\mathbf{r}^2 + \mathbf{p}^2\right] = \omega\left[\mathbf{a}^\dagger \cdot \mathbf{a} + 3/2\right]$$

1. Verify that the degeneration of the energy levels is higher than that which is implied by the $SU(3)$ rotational invariance.

2. Verify that the other conserved observables allow for the construction of three further $SU(2)$ algebras.

Hint: Remember that a two-dimensional harmonic oscillator admits an $SU(2)$ symmetry.

3. Write down some examples of perturbations that only partially removes the degeneration of the levels.

4. Explain why, although the various $SU(2)$'s allow for the construction of a complete algebra of $SU(3)$, it is not possible to find among the eigenstates of $\mathcal{H}$ bases for all the representations of $SU(3)$.

B.4. Statistical Mechanics of a Spin Chain

Consider a spin-chain, that is a one-dimensional lattice with a spin S_n placed at each site $x = n$ (with $n \in \mathbf{Z}$). Each spin S_n takes two possible values $+1$ and -1. The energy (Hamiltonian) of a configuration of spins is given by a nearest neighborhood interaction plus a magnetic term

$$H = J \sum_n (S_{n+1} - S_n)^2 + h \sum_n S_n.$$

1. Consider a finite chain of N spins and assume periodic boundary conditions $S_{N+1} \equiv S_1$. Compute the corresponding partition function Z_N for all values of the temperature T and magnetic field h.

2. Compute the free energy $\mathcal{F}$ in the thermodynamical limit $N \to \infty$ (infinite chain).

3. Assume $J > 0$. For the infinite chain, find for which temperatures T there is a spontaneous magnetization at zero magnetic field $h = 0$.

> Comment: By spontaneous magnetization we mean a spontaneous breaking of the $\mathbf{Z}_2$ symmetry of the $h = 0$ model signaled by a non–vanishing statistical expectaction value $\langle S_n \rangle|_{h=0}$.

4. Let $J < 0$. Describe the thermal state of the infinite chain in the limit $h \to 0$, $T \to 0$.

5.* In the infinite chain, compute the connected correlation function

$$\langle S_n\ S_m \rangle^{\text{conn}} = \langle S_n\ S_m \rangle - \langle S_n \rangle \langle S_m \rangle.$$

6.* Compute the correlation length ξ.

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Mathematical Physics Sector

Entrance examination 2002/2003 – October Session

The candidate is asked to solve at least one problem among the following.

1 Quantum Mechanics

A Harmonic oscillator in an electric field

Consider the one–dimensional quantum harmonic oscillator with Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$$

The particle has also a charge e. Switching on a constant electric field $\mathcal{E}$, the Hamiltonian gets an additional potential term and becomes

$$H'_- = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 - e\mathcal{E}x \tag{1.1}$$

1. Using the well-known expression of the energy spectrum of the harmonic oscillator, determine the spectrum of the new Hamiltonian.
2. Prove that the two operators H'_- and H'_+ (obtained by changing $\mathcal{E} \to -\mathcal{E}$ in eq.(1.1) are unitarily equivalent in $\mathcal{L}_2(-\infty, \infty)$.
3. Determine the value of $\mathcal{E}$ such that the energy of the first excited state $|\,\psi'_1\rangle$ of H'_- coincides with the energy of ground state $|\,\psi_0\rangle$ of H. Which observable permits to distinguish $|\,\psi'_1\rangle$ from $|\,\psi_0\rangle$?

4 Statistical Mechanics

One-dimensional random walk

At discrete units of time ($t = 1, 2, 3, \cdots$), a particle moves along a one dimensional lattice. Suppose it starts at the origin and that at each step it moves either one lattice site to the right or to the left, each with probability 1/2.

For $n \geq 1$, let u_n be the probability that the particle returns to the origin at time $t = n$ and p_n the probability that the first return to the origin occurs at $t = n$. For convenience we take $p_0 = 0$ and $u_0 = 1$. Clearly $p_n = u_n = 0$ whenever n is odd.

1. Show that for $n \geq 1$, the two probabilities are related as

$$u_n = p_0 u_n + p_1 u_{n-1} + p_2 u_{n-2} + \cdots + p_n u_0 \tag{4.1}$$

2. Prove that

$$u_{2n} = \frac{1}{2^{2n}} \binom{2n}{n} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} \tag{4.2}$$

3. Show that the particle will sooner or later return to its initial position, i.e.

$$p_2 + p_4 + p_6 + \cdots = 1$$

Hint. Introduce the generating functions

$$F(x) = \sum_{n=0}^{\infty} p_n x^n, \quad U(x) = \sum_{n=0}^{\infty} u_n x^n$$

and show that eq.(4.1) leads to the relation $U(x) = 1 + F(x)U(x)$. Use this relation and the u_{2n} given by eq.(4.2) to determine $F(x)$. It is useful to remind that

$$(1+x)^q = 1 + qx + \frac{q(q-1)}{2!}x^2 + \cdots + \frac{q(q-1)\ldots(q-k+1)}{k!}x^k + \cdots$$

Problem B. Classical Statistical Mechanics of a q-state Model

Let us consider an open one-dimensional lattice of N sites, with a fluctuating variable σ_i defined on each site i. The possible values assumed by σ_i are $\sigma_i = 1, 2, \ldots, q$ and the Hamiltonian of the model is given by

$$H(\sigma_1, \ldots, \sigma_N) = J \sum_{i=1}^{N-1} \delta(\sigma_i, \sigma_{i+1}) \ ,$$

where J is a real constant and

$$\delta(a,b) = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if } a \neq b \end{cases}$$

1. Under which transformations is the Hamiltonian invariant? Describe the ground state for $J > 0$ and for $J < 0$ and determine its degeneracy in the two cases.

2. Let us put the model in contact with a heat reservoir at temperature β^{-1}. Compute the partition function $Z_N(\beta J, q)$ defined by

$$Z_N(\beta J, q) = \sum_{\sigma_1=1}^{q} \cdots \sum_{\sigma_N=1}^{q} e^{-\beta H(\sigma_1, \ldots, \sigma_N)} \ .$$

   (*Hint.* Obtain a recursive equation $Z_N \to Z_{N-1}$ by using the mathematical identity $e^{x\delta(a,b)} = 1 + (e^x - 1)\delta(a,b)$).

3. Since $Z_N(\beta J, q)$ above computed is an analytic function of the variable q, the model results to be also defined for continous values of q and in particular in the "unphysical" region $q < 1$. Determine the occurrence of phase transitions for $J > 0$ and $J < 0$ and the critical temperature $\beta_c^{-1}(q)$ as a function of q.

Problem 1: Two coupled rotors

Consider two planar rotors, characterized by a fixed radius r, and angle ϕ_1 and ϕ_2, with Hamiltonian

$$h_i = \left(\frac{i\hbar}{r}\frac{\partial}{\partial\phi_i}\right)^2 \qquad\qquad i = 1, 2$$

1) Describe the energy spectrum and the eigenfunctions of each rotor, in the lack of interactions.

2) Consider next a coupling between the two rotors, of the form

$$V = -J\cos(\phi_1 - \phi_2) \qquad\qquad J > 0\,.$$

Taking now $H = h_1 + h_2 + V$, discuss the effect of the coupling on the low-energy part of the spectrum in the two opposite limits a) very small J; b) very large J.

Problem C. Stress-Energy Tensor of Quantum Field Theory between two planes

Consider two infinite planes which are parallel and at distance **a** apart along the z-axis. Suppose that a massless Quantum Field Theory is defined in the region between the two planes. The boundary conditions are such that the stress-energy tensor $T_{\mu,\nu}$ has a vacuum expectation value $t_{\mu,\nu}(t,\vec{x}) \equiv < 0 \mid T_{\mu,\nu}(t,\vec{x}) \mid 0 >$ different from zero. By virtue of the symmetry of this time-independent problem, $t_{\mu,\nu}$ must be written in terms of $g_{\mu,\nu}$ and tensors composed from the unit vector $\hat{z}_\mu = (0,0,0,1)$.

1. Write down the most general form of $t_{\mu,\nu}$ based upon based the considerations above.

2. Show that the conservation law $\partial^\mu T_{\mu,\nu}(t,\vec{x}) = 0$ and the condition that the trace of $T_{\mu,\nu}$ vanishes (which expresses the massless nature of the Quantum Field Theory) uniquely determine $t_{\mu,\nu}$ up to a constant. Use dimensional analysis to fix this constant up to a numerical factor in terms of a single parameter of the problem that bears a dimension.

3. Using the form of $t_{\mu,\nu}$ so obtained, compute the force per unit surface between the two planes.

Problem 1: Electron in a two–dimensional box

Consider an electron moving in two dimensions inside a generally rectangular box of potential $V(x, y) = 0$ when both $-a < x < a$ and $-b < y < b$
and $V(x, y) = \infty$ outside.

1. Assuming initially a square rigid box $a = b = a_0$, calculate the eigenfunctions and eigenvalues of the few lowest energy levels, specifying their respective symmetries and degeneracies.

2. Calculate the force F – due to the electron energy dependence upon the box size a_0 – exerted on the box walls by the electron, when it is

   (a) in the ground state

   (b) in the first excited state

3. Assume now the box to be elastically deformable, with a deformation energy

$$U = \frac{k}{2}[(a - a_0)^2 + (b - a_0)^2]$$

   for a general rectangular shape. Determine the nature and the (approximate) magnitude of the box deformation from the initial state $a = b = a_0$,

   (i) when the electron is in the ground state

   (ii) when the electron is in the first excited state.

   [Hint: assume the box deformation to be very small]

Problem 2: Motion of an electron in a harmonic trap

An electron is confined in an elipsoidal trap by the harmonic potential:

$$V(x, y, z) = \frac{m\omega_{\perp}^2}{2}(x^2 + y^2) + \frac{m\omega_{||}^2}{2}z^2.$$

1. Determine the ground state energy of the electron and sketch its energy levels (assume $\omega_{||} << \omega_{\perp}$).
2. Consider now turning on slowly a transverse electric field, $\Delta V = eEx$, up to a finite value, E_0, and calculate the ground state energy of the perturbed system.
3. How are the excitation energies of the system modified by the presence of the electric field ?
4. The electron is in the new ground state when the electric field is suddenly switched off. Compute the energy of the electron immediately after the electric field is removed.
5. Compute the time evolution of the average electron position after the electric field is removed.

Problem C. Reflection and Transmission Amplitudes

Consider a $(1+1)$ free massive boson theory but with a point of defect of coupling g at the origin. Its lagrangian density is given by

$$\mathcal{L} = \frac{1}{2}\left[(\partial_\mu \varphi)^2 - m^2 \varphi^2 - g\delta(x)\varphi^2 \right] \ . \tag{1}$$

The two–dimensional relativistic momenta may be expressed as $p_0 = m\cosh\beta$; $p_1 = m\sinh\beta$. As shown in the space–time diagram of Figure 1, the particle of the field φ propagates freely until it reaches the origin: here, the particle will be reflected and transmitted with amplitudes $R(\beta)$ and $T(\beta)$ respectively. Let $\varphi_\pm(x,t)$ be the field values for $x > 0$ and $x < 0$, respectively, i.e. $\varphi(x,t) = \theta(x)\varphi_+(x,t) + \theta(-x)\varphi_-(x,t)$, where $\theta(x)$ is the usual step–function.

1. Derive the boundary conditions for $\varphi_\pm(x,t)$ at the origin by imposing the equation of motion.

2. Using the free field expansion of $\varphi_\pm(x,t)$,

   $$\varphi_\pm(x,t) = \int d\beta \left[A_\pm(\beta) e^{-im(t\cosh\beta - x\sinh\beta)} + A_\pm^\dagger(\beta) e^{im(t\cosh\beta - x\sinh\beta)} \right]$$

   and considering the set of oscillators $A_\pm(\beta)$, impose the boundary conditions of point (1) to derive the explicit expressions for the reflection and transmission amplitudes $R(\beta)$ and $T(\beta)$.

   (*Hint.* It may be convenient to assume $A_-(-\beta) = 0$, see Figure 1)

   Will the answer be the same by considering the other oscillators $A_\pm^\dagger(\beta)$?

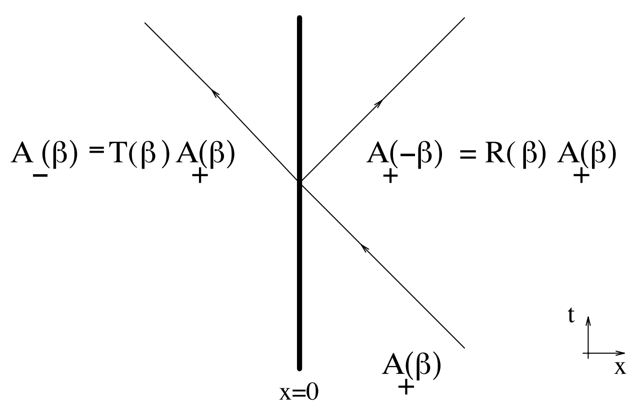

Figure 1.

Problem B. A Partition Function in Statistical Mechanics

Consider a quantum system whose energy spectrum (in appropriate unit) is given by

$$E_n = \log n \quad , \quad n = 1, 2, 3, \ldots \tag{1}$$

Once the system is in contact with a heat reservoir at temperature β^{-1}, its canonical partition function is given by

$$Z(\beta) = \sum_{n=1}^{\infty} \exp\left[-\beta E_n\right] \quad . \tag{2}$$

1. Discuss the range of β such that $Z(\beta)$ is well–defined and identify the finite value β_c of β for which there is a singularity.

2. Show that for $\beta \longrightarrow \beta_c$, the mean value of the energy of the system behaves as

$$\langle E \rangle \sim \frac{1}{\beta - \beta_c} \tag{3}$$

3. Prove that, for those values of β for which $Z(\beta)$ is well–defined, it holds the identity

$$Z(\beta) = \prod_k \frac{1}{1 - \frac{1}{p_k^{\beta}}} \tag{4}$$

   where the above product is on all prime numbers p_k.

   Hint Remember that for $|x| < 1$, $1/(1-x) = 1 + x + x^2 + x^3 + \cdots$.

4. Show that the existence of β_c implies that there are infinitely many primes.

Problem 1

Consider a zipper (see Figure) consisting of N bonds whose rightmost edge is held in place by an unspecified constraint. Due to thermal fluctuations the bonds can break starting from the free left edge. As a consequence, if the nth bond (from the left edge) is broken it implies that all bonds at the left are also broken (see figure). The breaking of a bond costs an energy equal to $\Delta > 0$.

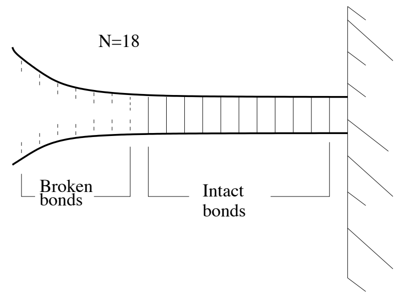

Figure 1: Schematic drawing of a zipper made of $N = 18$ bonds, 7 of which are broken.

Assuming that the possible states of each bond are only two (intact or broken):

1. calculate the partition function for the zipper,
2. calculate the temperature dependence of the system internal energy at low temperature,
3. calculate the number of broken bonds at very high temperature,
4. does the system have a singular behaviour in the limit $N \to \infty$?

Next consider the case where each *broken* bond can be in $g > 1$ configurations, each with the same energy Δ.

5. provide the partition function at low and high temperature,
6. calculate the number of broken bonds at low and high temperature,
7. does the system have a singular behaviour in the limit $N \to \infty$?

Problem 4. Eigenstates of one-dimensional potentials

You certainly remember that the gaussian wavefunction $\psi_0(x) = Ce^{-x^2/(2\sigma^2)}$ is the ground state solution of the harmonic potential.

1. Consider now the wavefunction

$$\psi_0(x) = Ce^{-\frac{1}{n}\left(\frac{x}{\sigma}\right)^n} , \tag{1}$$

   where σ is a quantity with dimension of a length, and n is a positive integer. For what class of integers n is ψ_0 an acceptable wavefunction? For each acceptable n, what is the corresponding potential $V_n(x)$ for which ψ_0 is an eigenstate with energy eigenvalue $E_0 = 0$? Plot the resulting potential for $n = 4$.

2. Generalize the previous construction, i.e., find out the general form of the potential $V(x)$ such that the following wavefunction

$$\psi_0(x) = Ce^{-\frac{W(x)}{W_0}} , \tag{2}$$

   is an eigenfuction with energy eiegenvalue $E_0 = 0$. (Here $W(x)$ is an arbitrary function, such that the resulting ψ_0 is normalizable, and W_0 is a constant with dimension of energy.) Can ψ_0 be an excited eigenstate? If not, why?

3. Consider now states of the form:

$$\psi(x) = P(x)e^{-\frac{1}{n}\left(\frac{x}{\sigma}\right)^n} , \tag{3}$$

   where $P(x)$ is a polynomial of degree ≥ 1 in x. Discuss why, on general grounds, ψ cannot be the ground state of a regular potential. By writing the Schrödinger equation explicitly, prove that ψ is a candidate *excited state* of the potential $V_n(x)$ found in point 1) only for $n = 2$.

Problem. Quantum particle on a unit circle

Consider a free quantum particle of mass m which lies on a circle of radius R. Its coordinate is q with $q \equiv q + 2\pi R$ and its lagrangian is given by

$$L = \frac{m}{2}\left(\frac{dq}{dt}\right)^2$$

1. Determine the energy eigenfunctions, the energy levels and their degeneracy.

2. Change the lagrangian according to

$$L = \frac{m}{2}\left(\frac{dq}{dt}\right)^2 - \hat{\theta}\,\frac{dq}{dt}$$

   Write the corresponding Hamiltonian and determine the new quantum energy levels. Study their degeneracy as a function of the dimensionless parameter $\theta = \hat{\theta}R/\hbar$ in the interval $0 \le \theta \le 1$.

3. Prove that the spectrum is invariant for $\theta \to \theta \pm k$, where k is any integer number.

4. Take $\theta = 1/3$ and suppose that at $t = 0$ system is in the physical state described by the wave function $\psi(q) = \cos q/R$. Determine the minimal time required to the system for coming back to the same physical state.