Fall 2006 - Entrance Examination: Statistical Physics

Solve one of the following problems (one well-solved problem is preferable to many partially addressed ones).
Write out solutions clearly and concisely. State each approximation used. Diagrams welcome. Number page, problem, and question clearly. Do not write your name on the problem sheet, but use the extra envelope.

Problem 1: Random system of spins

Consider a system of N spins σ_i ($i = 1, ..., N$) taking the value $\sigma_i = \pm 1$, with the Hamiltonian given by

$$H[\sigma_1, ..., \sigma_N] = -\sum_{i=1}^{N} h_i \sigma_i \tag{1}$$

The magnetic fields h_i are independent random variables, distributed according to a probability density $\rho(h)$, which is different from zero only on a finite interval $(-\alpha, \alpha)$. In the following, denote by $\langle \cdot \rangle$ the average over the distribution of the fields.

The system is in contact with a thermal bath at temperature T that leads to thermal equilibrium.

1. Compute the free-energy density for a given realization of the h_i and finite N. For all temperatures, prove its self-averaging property, namely the fact that approaching the thermodynamic limit $N \to \infty$ almost all the realizations of the magnetic fields give rise to a value of the free-energy close to its average over the field distribution.

2. Show that the square fluctuation of the free-energy is generically small, and compute it in the specific case of a field distribution

$$\rho(h) = p\delta(h-1) + (1-p)\delta(h-2) \tag{2}$$

   where $\delta(x)$ is the Dirac δ function.

3. Show that, differently from the free-energy, the relative mean square fluctuation of the partition function:

$$\frac{\langle Z^2 \rangle - \langle Z \rangle^2}{\langle Z \rangle^2} \tag{3}$$

   does not tend to zero for $N \to \infty$ (i.e. the partition function is not self-averaging). (Hint: use the fact that for any random variable x one has $\langle x^2 \rangle \geq \langle x \rangle^2$).

4. Suppose that the original Hamiltonian is modified by introducing a small interaction between the variables:

$$\hat{H}[\sigma_1, ..., \sigma_N] = -\sum_{i=1}^{N} h_i \sigma_i - \epsilon \sum_{i=1}^{N-1} \sigma_i \sigma_{i+1}. \tag{4}$$

   For the case of the distribution (2), compute the value of the average free-energy to the first order in the perturbation theory in ϵ.

Problem 2: Particles and Fields

By using natural units ($\hbar = c = 1$), consider the d-dimensional theory of a complex scalar field φ defined by the dimensionless action

$$A = \int d^d x\, \mathcal{L}(\varphi, \varphi^*) \tag{1}$$

where

$$\mathcal{L} = \partial_\mu \varphi\, \partial^\mu \varphi^* - \mu^2\, \varphi\, \varphi^* - g_1\, \varphi^3 - g_2\, (\varphi^*)^3 \tag{2}$$

1. Determine the symmetry transformations of φ and φ^* that leave the theory invariant for (i) $g_1 \neq g_2$ and (ii) $g_1 = g_2$.
2. From now on, assume that $g_1 = g_2 = g$. Let $|\, 0\,\rangle$ be the vacuum state of the theory, invariant under all the symmetries of the theory. Determine for which values of n and m the correlation functions
$$\langle\, 0 \mid \varphi(x_1) \ldots \varphi(x_n)\, \varphi^*(y_1) \ldots \varphi^*(y_m) \mid 0\,\rangle \tag{3}$$
are different from zero.
3. Write down the Feynman rules (propagator and vertices) associated to the theory. Let $A(p)$ and $A^*(p)$ be the particles, with momentum p, created by the fields φ and φ^*, respectively. Determine, at the lowest order in g, the amplitudes of the elastic scattering processes
$$A(p_1)\, A(p_2) \to A(p_3)\, A(p_4) \tag{4}$$
and
$$A(p_1)\, A^*(p_2) \to A(p_3)\, A^*(p_4) \tag{5}$$
4. From a dimensional point of view, the field φ behaves as Λ^α, where Λ is an energy scale. Find α and determine the dimension d_c of the space-time such that the coupling constant g is dimensionless. Can you comment on the physical meaning of d_c?

Problem 3: Stochastic one-dimensional walker

Consider a walker on a one-dimensional lattice with spacing a. At regular time intervals of duration Δt, the walker moves with probability $1/3$ by **two** lattice spacings to the right and with probability $2/3$ by **one** spacing to the left. The walker is in the origin, $x(0) = 0$ at time $t = 0$.

Assuming that all steps taken by the walker are independent.

1. Compute the mean displacement

   $$\langle x(n) \rangle$$

   and the mean square displacement

   $$\langle x^2(n) \rangle$$

   of the walker after n time steps. The mean is intended to be taken over many repetitions of the process.

2. Provide an expression for the probability $F_0(n)$ that the walker returns to the origin at a generic time n (not necessarely for the first time).

3. Based on the result of point 1, describe and provide an approximate formula for the probability distribution of the position of the walker at sufficiently large times, Prob[$x(n)$].

4. Consider now a different stochastic walker which starts from the origin at $t = 0$ and takes the first step with equal probability to the right or to the left by *one* lattice spacing. Subsequent steps of *one* lattice spacing have probability $1/2 + \epsilon$ to be in the same direction of the previous one, while probability $1/2 - \epsilon$ for the opposite one. Where will the walker be located on average when n time steps have elapsed? Will its square distance from the origin at a given n be larger or smaller than when $\epsilon = 0$? Assume that ϵ is a small positive quantity, so that only correlations between consecutive steps are significant.

Problem 4: Singlet ground state of a spin-1/2 chain

Consider three sites, $1, 2$, and 3, each occupied by a spin-1/2 operator $\vec{S}_i$ ($i = 1, 2, 3$) [here $\vec{S} = (1/2)\vec{\sigma}$ where $\vec{\sigma}$ are the Pauli matrices] and the three-site Hamiltonian

$$H_{1,2,3} \;=\; \frac{J}{4}(\vec{S}_1 + \vec{S}_2 + \vec{S}_3)^2 = \frac{J}{2}\left(\vec{S}_1 \cdot \vec{S}_2 + \vec{S}_2 \cdot \vec{S}_3 + \vec{S}_1 \cdot \vec{S}_3 + \frac{9}{8}\right) , \qquad (1)$$

with $J > 0$.

1 By simple considerations of addition of angular momenta, determine the minimum eigenvalue of $H_{1,2,3}$ and the corresponding eigenstates.

Consider now a ring with an *even* number N of sites, and a spin-1/2 operator $\vec{S}_i$ on each site $i = 1 \cdots N$, with the Hamiltonian

$$H_{\rm JK} = J \sum_{i=1}^{N} \vec{S}_i \cdot \vec{S}_{i+1} + K \sum_{i=1}^{N} \vec{S}_i \cdot \vec{S}_{i+2} , \qquad (2)$$

with $J > 0$ and $K > 0$. [Clearly, the ring geometry implies that one has to assume $\vec{S}_{N+1} = \vec{S}_1$ and $\vec{S}_{N+2} = \vec{S}_2$.]

2 Determine the value of K for which it is possible to rewrite $H_{\rm JK}$ in terms of three-site Hamiltonians as follows:

$$H_{\rm JK} = H_{1,2,3} + H_{2,3,4} + \cdots + H_{\rm N-1,N,1} + H_{\rm N,1,2} + \text{const} . \qquad (3)$$

3 Fix from now on K to the particular value, determined above, for which Eq. 3 holds true. Using the fact that *for this particular value of* K $H_{\rm JK}$ is a sum of positive definite terms (the three-site $H_{\rm i,i+1,i+2}$), of which one knows the minimum energy states, show that the ground state $|\Psi^{(N)}\rangle$ of the Hamiltonian in Eq. (3) can be written as a product of $N/2$ *singlets* as follows:

$$|\Psi^{(N)}\rangle = (1, 2)\,(3, 4) \cdots (N-1, N) , \qquad (4)$$

where

$$(i, i+1) = \frac{1}{\sqrt{2}} \left(|\uparrow\rangle_i |\downarrow\rangle_{i+1} - |\downarrow\rangle_i |\uparrow\rangle_{i+1}\right) .$$

4 Calculate the ground state energy of $|\Psi^{(N)}\rangle$. Can you provide another simple ground state of $H_{\rm JK}$ closely related to $|\Psi^{(N)}\rangle$?