Fall 2007 - Entrance Examination: Statistical Physics

Solve one of the following problems (one well-solved problem is preferable to many partially addressed ones).

Write out solutions clearly and concisely. State each approximation used. Diagrams welcome. Number page, problem, and question clearly.

Do not write your name on any problem sheet, but use the extra envelope.

Problem 1. Elastic properties of a Gaussian chain

Consider a one-dimensional Gaussian chain constituted by N beads connected by harmonic springs of elastic constant k. The Hamiltonian of the system is:

$$\mathcal{H} = \frac{k}{2} \sum_{i=1}^{N-1} (x_{i+1} - x_i)^2 \tag{1}$$

where x_i denotes the position of the ith bead.

Assume that the system is in thermal equilibrium at temperature T. Moreover, invoking the translation invariance of the Hamiltonian, disregard in the solution of the exercise the degree of freedom associated to the center of mass of the chain. Indicating with $R_{ee} \equiv x_N - x_1$ the end-to-end distance of the chain:

1. Calculate the canonical expectation values: $\langle R_{ee} \rangle$ and $\langle R_{ee}^2 \rangle$.
2. Consider now the application of a tensile force, f, at the ends of the chain:

$$\mathcal{H} = \frac{k}{2} \sum_{i=1}^{N-1} (x_{i+1} - x_i)^2 - (x_N - x_1)\, f. \tag{2}$$

   Calculate how the system free energy F varies as a function of f (at fixed temperature).

3. Calculate the canonical expectation value of the end to end distance and its square fluctuation, $\langle R_{ee} \rangle$ and $\delta^2 R_{ee} \equiv \langle R_{ee}^2 \rangle - \langle R_{ee} \rangle^2$, as a function of f.
4. Setting $f = 0$ write the partition function, $\mathcal{Z}(R_{ee})$, restricted to the configurations with a specific value of R_{ee}. Calculate the associated free energy profile, $F(R_{ee})$ and derive the average force that is necessary to apply to mantain the end-to-end separation equal to a preassigned value, $\bar{R}_{ee}$.

Useful formulas:

$$\int_{-\infty}^{+\infty} dx\, e^{-\gamma x^2} = \sqrt{\frac{\pi}{\gamma}}$$

$$\delta(a) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dp\, e^{iap}$$

Problem 2. Spins with Competing Couplings

Consider the Hamiltonian of a $1D$ Ising spin chain with free boundary conditions and total number of sites equal to N, and with two competing interactions

$$H = -J_1 \sum_i S_i S_{i+1} + J_2 \sum_i S_i S_{i+2}, \tag{1}$$

where J_1 and J_2 are both positive quantities. S_i takes values $S_i = \pm 1$. Let us define $\chi = J_2/J_1$.

1. Discuss qualitatively the ground-state configuration for $\chi \ll 1$ and $\chi \gg 1$.

2. Let us call *ordered* the configuration where all spins are alligned and *domain wall* the configuration in which all the spins after a given site are flipped. Show that, for χ smaller than a critical value χ_c, the ordered configuration has lower energy than the domain wall; determine the value of χ_c where the two configurations are degenerate.

3. At $\chi = \chi_c$, the ground-state is highly degenerate (explain why). Let a_N be the number of degenerate ground-states of a chain of length N, with

   $$a_N = b_N + c_N \tag{2}$$

   where b_N and c_N are the number of ground-state configurations in which the last two spins are parallel or antiparallel, respectively. By using eq. (2) and the above definitions of b_N and c_N, derive a recursion relation for a_N and compute the entropy per spin in the limit $N \to \infty$.
   Hint: once the recursive equation is obtained, set $a_N \sim \phi^N$ to solve it.

4. By using the identity $S_i^2 = 1$, show that the Hamiltonian (1) is equivalent to an antiferromagnetic Ising chain in a magnetic field.

Problem 3. Field Theory

Consider the theory of a scalar field defined by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) - V(\varphi)$$

in a (1+1)-dimensional Minkowski space-time, with x^0 as time and $x^1 \equiv x$ as space coordinates.

1. Show that the static solutions of the equation of motion satisfy the equation

$$\frac{1}{2}\left(\frac{d\varphi}{dx}\right)^2 - V \;=\; \text{constant}$$

   (Hint: observe the formal analogy of the static equation of motion with the motion of a non-relativistic particle of unit mass for which φ denotes the coordinate and x the time).

2. Consider the case in which the potential V coincides with

$$V_0 = \lambda(\varphi^2 - \varphi_0^2)^2$$

   (λ and φ_0 are positive constants). Determine the static solution $\varphi_{kink}(x)$ satisfying

$$\varphi_{kink}(\pm\infty) = \pm\varphi_0 \quad ; \quad \varphi'_{kink}(\pm\infty) = 0$$

   where $\varphi' \equiv d\varphi/dx$.

3. (a) Recalling that the energy-momentum tensor is given by

$$T^{\mu\nu}(x) = (\partial^\mu \varphi)\left(\frac{\partial \mathcal{L}}{\partial(\partial^\nu \varphi)}\right) - g^{\mu\nu}\mathcal{L}$$

   determine the value of x for which the energy density $T^{00}(x)$ of the configuration φ_{kink} takes its maximum value and also compute the total energy of the configuration φ_{kink}. (b) Show that the topological current $J^\mu = \varepsilon^{\mu\nu}\partial_\nu\varphi$ ($\varepsilon^{\mu\nu}$ is the unit antisymmetric tensor) is conserved and determine the topological charge Q of φ_{kink}.

4. Consider now the perturbed potential $V = V_0 + h\varphi$.

   (a) Do exist configurations of the unperturbed system with topological charge $Q \neq 0$ that have finite energy when $h \neq 0$? (b) Discuss the expectation value $\langle\varphi\rangle$ in the two cases $h = 0$ and $h \neq 0$.

Useful integrals: $\int \frac{dx}{1-x^2} = \operatorname{Arcth} x\,,$ $\qquad \int_0^\infty \frac{dx}{\cosh^4 x} = \frac{2}{3}$

Problem 4. A simple model for the equilibrium of a gas

Consider a gas of N particles and let them be divided in two containers A and B of equal volume. At each time t one particle is chosen at random and moved from its container to the other one. This process is repeated for an indefinite number of time steps $t = 0, 1, \ldots$.

Let $X(t)$ be the number of particles in container A at time step t.

1. Write down the transition probabilities

$$p_{i,j} = \text{Prob}\{X(t+1) = j | X(t) = i\}$$

(this is the probability to find j particles in A at time $t+1$ given that there were i in A at time t). Using these, write down the equation for the evolution of

$$\mu_k(t) = \text{Prob}\{X(t) = k\}.$$

2. Consider the possible processes from a state with $X(t_0) = k_0$ particles in box A and $X(t_1) = k_1$ particles in box A at a later time t_1. Argue that the probability $\mu_k(t)$ of any state k for $t \to \infty$ is strictly positive (i.e. that the process is ergodic).

3. Using the fact that this process is reversible, find the stationary state distribution

$$\mu_k^* = \lim_{t\to\infty} \mu_k(t)$$

and compute the fluctuations $\langle (N_A - N_B)^2 \rangle$ of the difference in the number of particles inthe two boxes.

(Hint: A process is reversible if it satisfies detailed balance, i.e. if the stationary state distribution satisfies the detailed balance condition $\mu_k^* p_{k,j} = \mu_j^* p_{j,k}$.)

4. Consider now the action of a daemon, similar to Maxwell's one. This modifies the process described above in the following way: When particles are selected to move from A to B the daemon impedes the transition with probability λ. The daemon has no effect on particles moving from B to A. In other words,

$$p^{(\lambda)}_{k,k-1} = (1-\lambda) p^{(0)}_{k,k-1}, \qquad p^{(\lambda)}_{k,k} = \lambda p^{(0)}_{k,k-1}, \qquad p^{(\lambda)}_{k,j} = p^{(0)}_{k,j} \ \text{ for } j \neq k, k-1$$

where $p^{(0)}_{k,j}$ are the original transition probabilities (i.e. for $\lambda = 0$) computed in point 1) above.

   - Compute the modified stationary state distribution $\mu^*_{k,\lambda}$, the average number $N_A = \langle k \rangle$ of particles in container A and its standard deviation $\sigma_a = \sqrt{\langle (k - \langle k \rangle)^2}$.
   - Note that, with $\lambda = 0$, the model describes an ideal gas of non-interacting particles, distributed in two boxes of equal volumes ($V_A = V_B = V/2$), in mechanical ($P_a = P_B = P$) and thermal ($T_A = T_B = T$) equilibrium. This equilibrium is perturbed by the daemon when $\lambda > 0$.

     Compute the work done by the daemon when *i)* the volume $V_A = V/2$ is kept constant in box A and the pressure $P_B = P$ is kept constant in box B. *ii)* The pressure $P_A = P_B = P$ is kept constant in both boxes A and B.

Problem 5. A quantum particle in a 2d harmonic potential

Consider the quantum problem of an electron confined in two dimensions (XY-plane) and subject to a harmonic potential:

$$H_0 = \frac{1}{2m}\left(P_x^2 + P_y^2\right) + \frac{1}{2}m\omega^2\left(X^2 + Y^2\right) \ . \tag{1}$$

1. Write down the eigenvalue spectrum of H_0, specifying the degeneracy of each level. Is H_0 a complete set of observables for this problem? What other operators, commuting with H_0, can be chosen to uniquely specify the states of the system?

2. Consider now adding to H_0 a term proportional to the angular momentum $\hbar L_z = XP_y - YP_x$:
$$H = H_0 - \hbar\Omega L_z \ . \tag{2}$$
Write down the expression of both H_0 and L_z in terms of the bosonic operators $a_x = (x + ip_x)/\sqrt{2}$ and $a_y = (y + ip_y)/\sqrt{2}$, where $(p_x, p_y) = (l/\hbar)(P_x, P_y)$ and $(x, y) = (X, Y)/l$ are dimensionless variables expressed in terms of the oscillator length $l = \sqrt{\hbar/(m\omega)}$. What are the correct linear combinations of a_x and a_y that diagonalize both H_0 and L_z? [**Hint**: Try to write L_z as $L_z = \left(a_+^\dagger a_+ - a_-^\dagger a_-\right)$ where $a_\pm$ are the required linear combinations.] What is the spectrum of H? Give an interpretation of the operators $a_\pm$.

3. Show that a Hamiltonian of the form (2) is obtained by considering an electron in presence of a magnetic field directed along the Z-axis (neglecting spin):
$$H_B = \frac{1}{2m}(\mathbf{P} + \frac{e}{c}\mathbf{A})^2 + \frac{1}{2}m\omega_d^2\mathbf{R}^2 \ , \tag{3}$$
where $\mathbf{A} = (B/2)(Y, -X, 0)$ is the vector potential in the symmetric gauge, $\mathbf{R} = (X, Y)$ and $\mathbf{P} = (P_x, P_y)$. Discuss what relationship ω and Ω in Eqs. (1-2) have with ω_d and the magnetic field B in H_B.

4. Show what happens to the degeneracy of the ground state of H_B in the limit in which $B \to \infty$ (or $\omega_d \to 0$). Construct the wavefunctions of the ground state in such a limit. [**Hint**: Use the fact that angular momentum quanta are created by $a_+^\dagger = z/2 - \partial/\partial z^*$, where $z = x + iy$ and $\partial/\partial z^* = (\partial/\partial x + i\partial/\partial y)/2$, upon acting on the ground state, expressed in terms of z and z^*.]