Fall 2008 - Entrance Examination: Statistical Physics

Solve one of the following problems (one well-solved problem is preferable to many partially addressed ones).

Write out solution clearly and concisely. State each approximation used. Diagrams welcome. Number page, problem and question clearly.

Do not write your name on any problem sheet, but use the extra envelope.

Problem 1. Negative temperature

Consider a system of N free classical particles. The energy of each particle can take only two distinct values, that we choose to be 0 and ϵ, with $\epsilon > 0$. Let's denote by n_0 and n_1 the occupation numbers of the energy levels 0 and ϵ respectively. In the following we consider the statistical mechanics of such a system in the microcanonical ensemble, i.e. the statistical ensemble in which the total energy E is a conserved quantity.

1. Find the entropy $S(E)$ of the system as a function of the energy E.

   Hint. Assume that all quantities entering the entropy are large numbers and use the Stirling approximation

   $$\log x! \simeq x \log x$$

   for the factorial of a large number x.

2. Determine the temperature T of the system as a function of E and show that it can be negative. Explain why and how this happens by computing the mean value of n_0 and n_1 as functions of T.

3. Explain why the values of the entropy at $E = 0$ and at $E_{max} = N\epsilon$ make negative temperatures unavoidable for the system considered above. Argue that instead negative temperatures do not arise in systems of particles with an infinite set of levels with increasing energies

   $$0, \epsilon_1, \epsilon_2, \ldots, \epsilon_n, \ldots$$

   where

   $$\lim_{n\to\infty} \epsilon_n = +\infty$$

4. In which direction does the heat flow if a system of negative temperature is put in contact with a system of positive temperature?

Problem 2. Spin-orbit coupling

Consider the Hamiltonian

$$H = -\frac{\hbar^2 \vec{\nabla}^2}{2m} + \alpha(\vec{E} \times (-i\vec{\nabla})) \cdot \vec{\sigma},$$

where $\vec{E}$ is an electric field constant in space and time, and σ are Pauli matrices. This Hamiltonian describes free electrons subject to a spin-orbit interaction with coupling strength α.

1) What are the symmetries of the Hamiltonian. Is the system invariant under time reversal ?

2) Without loss of generality, let us consider the electric field to be of strength E_z and pointing in the z direction. Suppose that at time $t = 0$ an electron is injected in a state $\mid \Psi\rangle$ with momentum k_x in the x direction and spin pointing in the z direction. Find the time evolution of the spin projection.

3) Let us again consider an electric field pointing in the z direction, and an electron confined in the $x - y$ plane. What is the expression for the x and y components of the velocity operator, $\hat{v}_x$ and $\hat{v}_y$? Compute $\langle\Psi \mid \hat{v}_x \mid \Psi\rangle$ and $\langle\Psi \mid \hat{v}_y \mid \Psi\rangle$, with $|\Psi\rangle$ specified at point 2), and give a physical interpretation to the result obtained.

4) In the situation above, find the eigenvalues and eigenstates of H. Plot the spectrum vs. $k = \sqrt{k_x^2 + k_y^2}$.

Problem 3. Condensation phase transition in a system of interacting particles

Consider a system of classical particles on a d dimensional lattice of N sites. Let n_i be the number of particles on site $i = 1, \ldots, N$. Particle interact only with particles on the same site, with an attractive interaction given by the Hamiltonian

$$H\{n_i\} = \sum_{i=1}^{N} \log(1 + n_i).$$

1. Write down the grand canonical partition function $\mathcal{Z}(\beta, \mu)$ for the system at inverse temperature β and chemical potential μ and derive the expression for the density
$$\rho = -\frac{\partial}{\partial \mu}\frac{1}{\beta} \log \mathcal{Z}$$
as a function of β and μ. Draw a qualitative plot of ρ as a function of μ at high temperatures and at low temperatures.

2. Now consider the system at finite density $\rho = M/N$, where M is the number of particles. Find the ground state as a function of the density.

3. Discuss the behavior of the system at finite density ρ. Show that, for $\beta > 2$, there is a critical density ρ_c above which a finite fraction of particles condense in a single site. Find the expression of ρ_c.

4. Do the results depend on the dimensionality of the system? Why?

Problem 4. Correlations in field theory

Consider a relativistic quantum theory of spinless particles of mass m in (2+1)-dimensional space-time and let $\varphi(x)$ be a real scalar field, with $x = (t, \mathbf{x})$ denoting a point in space-time.

1. Assume the Lorentz invariant normalization $\langle p'|p\rangle = e\,\delta(\mathbf{p}' - \mathbf{p})$ for the single particle states, with $p = (e, \mathbf{p})$ denoting the energy-momentum of a particle. Consider $f_p(x) = \langle 0|\varphi(x)|p\rangle$, i.e. the matrix element of the field between the vacuum state and the single particle state. Determine $f_p(x)/f_p(0)$ and explain why $f_p(0)$ is a constant.

2. From now on suppose that $f_p(x)$ and its conjugate are the only non-vanishing matrix elements of $\varphi(x)$ on particle states. Determine up to an overall constant the correlator $G(x) = \langle 0|\varphi(x)\varphi(0)|0\rangle$ in the euclidean configuration $x = (i\tau, 0)$.

   Hint: deduce the integration measure in momentum space from the state normalization given above.

3. Use the previous result to write down $G(x)$ for the generic euclidean configuration $x = (i\tau, \mathbf{x})$ and determine the second moment correlation length

$$\xi = \left(\frac{\int d^3x\, |x|^2\, G(x)}{6 \int d^3x\, G(x)} \right)^{1/2},$$

   where $|x| = \sqrt{x^\mu x_\mu}$ and the integrations are performed over the three-dimensional euclidean space.

4. For the case $m = 0$, use a dimensional argument to generalize the result for $G(x)$ to the d-dimensional euclidean space.

Problem 5. Statistical properties of relativistic particles

Consider a three-dimensional **classical** gas of N indistinguishable particles in a volume V having ultrarelativistic energies, i.e. their energy-momentum relation is given by

$$\epsilon(\vec{p}) = cp$$

where $\vec{p}$ is the particle momentum and c is the speed of light.

1. Compute the partition function for this system and determine the equation of state, i.e., the relation among the pressure, the volume and the temperature.

2. Compute the total average energy and comment on the result. If the dimension of the space is D, how the obtained result is modified?

3. Consider now the effect of the bosonic statistic, i.e., assume that the particles are bosons: writing the Hamiltonian as

$$H = \sum_{\vec{p}} \epsilon(\vec{p}) \alpha_{\vec{p}}^{\dagger} \alpha_{\vec{p}} - \mu \sum_{\vec{p}} \alpha_{\vec{p}}^{\dagger} \alpha_{\vec{p}}$$

   (where $\alpha_{\vec{p}}$ is the operator destroying a particle with momentum $\vec{p}$), write a relation between the chemical potential μ and the temperature T.

4. The Bose-Einstein condensation critical temperature T_c is defined by the condition $\mu(T_c) = 0$: do these massless relativistic particles condense?