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Fall 2009 - Entrance Examination: Statistical Physics

Solve one of the following problems (one well-solved problem is preferable to many partially addressed ones).

Write out solution clearly and concisely. State each approximation used. Diagrams welcome. Number page, problem and question clearly.

Do not write your name on any problem sheet, but use the extra envelope.

Problem 1. Time Evolution in the Anharmonic Oscillator

Consider the one-dimensional harmonic oscillator with Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 \,,$$

where $[x, p] = i\hbar$. With the notation $X = x\sqrt{m\omega/\hbar}$, $P = p/\sqrt{m\omega\hbar}$, introduce the annihilation and creation operators

$$a = \frac{1}{\sqrt{2}}(X + iP) \quad , \quad a^\dagger = \frac{1}{\sqrt{2}}(X - iP)$$

We recall that $H = \hbar\omega(N + 1/2)$, with $N = a^\dagger\, a$, $N|n\rangle = n\,|n\rangle$ and $\langle n|m\rangle = \delta_{n,m}$.

1. Evaluate the commutator $[N, a]$ and prove that $a|n\rangle = \sqrt{n}|n-1\rangle$ within a phase factor that can be put equal to 1.

2. Consider an arbitrary complex number α. Using a series expansion of the states on the $|n\rangle$ basis, determine the normalized eigenstate $|\alpha\rangle$ of the annihilation operator a with eigenvalue α, i.e. $a|\alpha\rangle = \alpha\,|\alpha\rangle$.

3. Compute the expectation values of H, x and p on the state $|\alpha\rangle$ and show that the square deviations Δx and Δp on this state satisfy $\Delta x\,\Delta p = \hbar/2$, independently of α.

4. Suppose that for time $t > 0$ the Hamiltonian is just given by

$$W \;=\; \hbar g (a^\dagger\, a)^2 \quad , \qquad g > 0 \;.$$

   With the initial condition $\psi(t = 0) = |\alpha\rangle$, write the expression of the state $\psi(t)$ and show how this state simplifies in the particular cases $t = 2\pi/g$ and $t = \pi/g$. Prove that for $t = \pi/(2g)$ the state $\psi(t)$ can be written as

$$\psi(t) \;=\; \frac{1}{\sqrt{2}}\left(e^{-i\pi/4}|\alpha\rangle + e^{i\pi/4}|-\alpha\rangle\right) \;.$$

   When α is purely imaginary, comment on the nature of this state, in particular if it has a classical realization.

Problem 2. Probability distributions in a gas

A vessel of volume V contains N non-interacting gas molecules. Let n be the number of molecules in a part of the vessel of volume v. Considering that the probability of finding a certain molecule in v is equal to v/V in the thermal equilibrium of the system

1. find the probability distribution $f(n)$ of the number n

2. Compute the generating function

$$F(z) = \sum_{n=0}^{N} f(n) z^n$$

   Interprete this quantity as a fictitious grand-canonical partition function with fugacity z and discuss why, a-priori, one should expect that $F(z)$ has only one zero of order N at $z = 1 - V/v$.

   Hint. Compute first $F(z)$ for $N = 1$ and then set up a recursive argument.

3. Use the generating function to calculate both the mean value $\overline{n} \equiv \langle n \rangle$ and the variance $\langle (n - \langle n \rangle)^2 \rangle$

4. (a) Show that when N and n are both large, for n close to $\overline{n}$, $f(n)$ becomes a gaussian distribution

   Hint. Recall the Stirling formula for the factorial

$$\log N! \simeq N(\log N - 1)$$

   (b) Using the generating function $F(z)$, show that if $v/V \to 0$ with $\overline{n}$ = constant, $f(n)$ approaches instead the Poisson distribution

$$f(n) = e^{-\overline{n}} \frac{(\overline{n})^n}{n!}$$

Problem 3. Ising percolation

Consider an infinite regular two-dimensional lattice with spin variables $\sigma_i = \pm 1$ at each site. If, for a given configuration, we draw a link between the nearest neighbors with *positive* spin, we obtain a number of connected sets of positive spins that we call clusters.

Let P be the probability that, when averaging over all configurations, the site in the origin belongs to an infinite cluster. The transition having P as order parameter is called percolation transition.

1. Suppose that the spins do not interact and that each of them has a probability p to be positive. There exists a positive value p_c such that $P \neq 0$ for $p > p_c$. Do you expect p_c to be larger for the square lattice or for the triangular lattice? Motivate your answer.

2. Consider now the case in which the weight of each configuration is $e^{-\mathcal{H}}$, where

$$\mathcal{H} = -\frac{1}{T} \sum_{\langle ij \rangle} \sigma_i \sigma_j - H \sum_i \sigma_i$$

   is the Ising reduced Hamiltonian with nearest neighbor interaction and $T \geq 0$. Argue that for any value of T there is a percolation transition in the (T, H) plane. Determine the value of H for which the percolation transition occurs at $T = 0$. Which is the order of the percolation transition at $T = 0$?

3. Determine the value of H for which the percolation transition occurs at $T = \infty$. Is it possible that, for T large enough, the percolation transition occurs at $H < 0$?

4. Suppose that each site with positive spin is colored in blue with probability $1 - e^{-1/T}$. Consider the clusters made of *blue* sites and determine the location of the percolation transition, if it exits, at $T = 0$, $T = \infty$ and $H = +\infty$. Draw qualitatively the percolation transition line for the blue clusters in the (T, H) plane.

Problem 4. Fermionic Chain in an Alternating Field

Consider a one-dimensional chain of (spinless) fermions made of $N \gg 1$ sites (with N an even integer) and described by the Hamiltonian:

$$H_0 = -t \sum_{i=1}^{N} \left(c_i^\dagger c_{i+1} + h.c. \right) \equiv \sum_{i,j=1}^{N} t_{ij} c_i^\dagger c_j \tag{1}$$

where $i = 1, \cdots, N$ denotes the sites and the c_i are fermionic operators with anti-commutation relations $\{c_i, c_j\} = 0$, $\left\{c_i, c_j^\dagger\right\} = \delta_{ij}$. Assume periodic boundary conditions ($c_{N+1} = c_1$). The number of fermions is N_F: $N_F = N/2$ corresponds to the half filling case, i.e. half of the available states are filled and half are empty.

1. Compute the eigenvalues and the normalized eigenvectors $\psi_\alpha(i)$ of the matrix t_{ij}, where α is the quantity that labels the eigenvectors.

2. Introduce the operators $d_\alpha = \sum_{i=1}^{N} c_i \psi_\alpha(i)$. Determine the anti-commutation relations $\left\{d_\alpha, d_\beta^\dagger\right\}$ and express the Hamiltonian (1) in terms of the operators d_α. Compute the energy eigenvalues and the density of states $\rho(E)$.

   (*Hint: to check that $\rho(E)$ is properly normalized, use $\int dx/\sqrt{1-x^2} = \arcsin x$*).

3. Compute the Fermi energy E_F as a function of N_F/N and determine its value at half-filling.

4. Suppose that an alternating field is added: the new Hamiltonian reads $H = H_0 + H_{alt}$, where

   $$H_{alt} = \sum_{i=1}^{N} \epsilon_i c_i^\dagger c_i, \tag{2}$$

   with $\epsilon_i = (-1)^i \delta$. Proceed as in the item (*1*) above: write H as $H \equiv \sum_{i,j=1}^{N} \tilde{t}_{ij} c_i^\dagger c_j$ and compute the new energy spectrum.

   (*Hint: write the eigenfunctions $\psi_\alpha(i)$ of $\tilde{t}_{ij}$ in the form $u_\alpha(i)\psi_\alpha^{(0)}(i)$, where $\psi_\alpha^{(0)}(i)$ are the eigenfunctions in the unperturbed $\delta = 0$ case*).

   What happens at half-filling in this case?