Fall 2010 - Entrance Examination: Statistical Physics

Solve one of the following problems (one well-solved problem is preferable to many partially addressed ones).

Write out the solution clearly and concisely. State each approximation used. Diagrams and plots are welcomed. Number pages, problems and questions clearly.

Do not write your name on any problem or solution sheet but use instead the tag in the extra envelope.

Problem 1. Quantum tops

Consider a quantum top with Hamiltonian

$$H(L) = \alpha_x L_x^2 + \alpha_y L_y^2 + \alpha_z L_z^2\,,$$

where L_i are the angular momentum operators with commutation relations $[L_k, L_l] = i\epsilon_{klm} L_m$ ($\hbar = 1$ in what follows) and the α's are related to the moments of inertia. We denote by $|l, m\rangle$ the simultaneous eigenvectors of $\vec{L}^2$ and L_z.

Consider first the case $\alpha_x = \alpha_y \neq \alpha_z$.

1. Calculate the eigenvectors of the Hamiltonian, the corresponding eigenvalues and degeneracies. What is the expectation value of $L_x + L_y + L_z$ on each eigenstate of the Hamiltonian?

2. The state of the top at time $t = 0$ is $|l = 3, m = 0\rangle$. What is the probability to obtain the value $\hbar$ in a measurement of L_z at time $t = 4\pi I_1/\hbar$?

Consider now the generic case $\alpha_x \neq \alpha_y \neq \alpha_z \neq \alpha_x$

3. Calculate the eigenvalues of the Hamiltonian when $l = 1$.
   (The matrix elements $\langle 1, m|L_x|1, m-1\rangle = \frac{1}{2}\sqrt{(1+m)(2-m)}$ might be useful.)

Consider now two interacting fully symmetric (i.e., $\alpha_x = \alpha_y = \alpha_z \equiv \alpha$) tops 1 and 2 with Hamiltonian

$$H = H_1 + H_2 + \lambda \vec{L}_1 \cdot \vec{L}_2\ ,$$

where L_1 and L_2 are the angular momentum operators of the two tops and $H_1 = H(L_1)$, $H_2 = H(L_2)$.

4. Does H commute with $\vec{L}_1$ and $\vec{L}_2$ separately? Why? Exploit symmetries in order to determine $[H, \vec{L}_1 + \vec{L}_2]$.

   Assume now that the tops 1 and 2 have spin 1/2. Calculate the energy levels of this composite system and their degeneracy. Which term would you add to the Hamiltonian in order to lift the residual degeneracy? Why?

Problem 2. Random walk with shrinking steps

A random walker in one dimension starts at $x = 0$ and takes consecutive steps to the left or to the right with equal probability. The first step has unit length and each of the following ones has a length which is a factor $\lambda > 0$ of the previous one. Indicate by x_N the position of the walker after N steps.

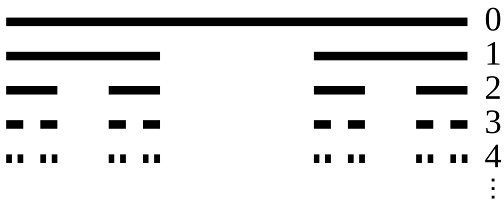
1. Determine the variance $\langle x_N^2 \rangle$ and the maximal distance $x_N^{(\max)}$ that the walker can travel in N steps. Discuss their behaviors for large N as functions of λ.

2. Indicate by $P_\lambda(x, N)$ the probability distribution function of the random walker after N steps. Determine the relation between $P_\lambda(x, N+1)$ and $P_\lambda(x, N)$ and use it in order to calculate explicitly the Fourier transform $\hat{P}_\lambda(k, N) \equiv \int_{-\infty}^{+\infty} \mathrm{d}x \, \mathrm{e}^{ikx} P_\lambda(x, N)$.

In what follows consider the case $\lambda < 1$ and indicate by $P_\lambda(x)$ the probability distribution function of the end point of the random walk.

3. Assuming that you know the position x_n of the walker after n steps, determine the closed interval $I_n(x_n)$ of minimal length (centered around x_n) which *surely* contains the endpoint x_∞ of that walk. Use this result in order to show that for $\lambda < 1/2$ the support of $P_\lambda(x)$ is a Cantor set.

   (*Reminder: A Cantor set is the limit of the sequence of intervals which is obtained from one single initial interval by iteratively removing an open central subinterval from each interval.*)

4. Using the result of point 2, determine explicitly the distribution function $P_\lambda(x) = \lim_{N\to\infty} P_\lambda(x, N)$ for $\lambda = 1/2$.

   (*Hint:* $\cos\alpha = \sin(2\alpha)/(2\sin\alpha)$.)

Problem 3. Rabi pulse for trapped particles

Rabi pulses are used in quantum optics to transfer the wavefunction of a system from a certain level to another level. In this problem we first focus on the simplified case of a two-state system with Hamiltonian $H = H_0 + V(t)$, where $H_0 = (\hbar\omega/2)\,\sigma_z$ and $V(t) = (\hbar\nu/2)\sin(\Omega t)\,\sigma_x$ is the time-dependent perturbation and σ_α $(\alpha = x, y, z)$ are the usual Pauli matrices. The eigenfunctions of H_0 will be denoted by $\psi_\pm$.

1. Assume that the system is initially described by the wavefunction ψ_-. On physical ground, for which values of Ω do you expect to be able to induce most efficiently the transition to the state ψ_+?

2. Under the same assumption as point 1 for the initial wavefunction, write down the evolution equations for the coefficients $C_\pm(t)$ (in the basis $\{\psi_\pm\}$) of the time-dependent wavefunction $\psi(t)$. The *fidelity* of the pulse is defined as $F(t) = |\langle\psi(t)|\psi_+\rangle|$. The perfect transfer corresponds to $F = 1$. Calculate $F(t)$ for $\omega = \Omega$, assuming that one can neglect oscillatory terms in the differential equations. Plot F as a function of time: At which times is the fidelity equal to 1?

Consider now a quantum particle in one dimension, trapped in an harmonic potential $m\omega^2 x^2/2$ and subject to a sinusoidal perturbation $V(t) = \lambda\sin(\Omega t)\,\mathcal{O}$, where $\mathcal{O}$ is a generic operator.

3. Assume the particle to be at time $t = 0$ in the ground-state $\psi_0(x)$ of the harmonic trap and that we want to transfer it to the first excited state $\psi_1(x)$ of the trap. The fidelity of the pulse V is then defined as $F = |\langle\psi(t)|\psi_1\rangle|$. Which operator $\mathcal{O}$ would you suggest to use to achieve the maximum fidelity? Why?

4. Write down and discuss the properties of the equations for the coefficients of the wavefunction $\psi(x,t)$ on the harmonic oscillator basis. Restricting the dynamics to the space spanned by ψ_0 and ψ_1, calculate and plot the fidelity as a function of time.

Problem 4. Quantum quench

Consider a chain of N coupled harmonic oscillators with Hamiltonian

$$H(m) \;=\; \frac{1}{2}\sum_{r=1}^{N}\left[\pi_r^2 + m^2\phi_r^2 + (\phi_{r+1}-\phi_r)^2\right]\;;$$

m is a "mass" parameter whereas ϕ_r and π_r are, respectively, the position and momentum operators of the r-th oscillator, with equal-time commutation relations

$$[\phi_r,\pi_n] = i\delta_{rn} \quad , \quad [\phi_r,\phi_n] = [\pi_r,\pi_n] = 0\,.$$

Assume periodic boundary conditions for the chain.

1. Determine the operators A_k, $A_k^\dagger$ and the function Ω_k which bring the Hamiltonian in the diagonal form $H(m) = \sum_k \Omega_k A_k^\dagger A_k$.

   (Hint: introduce Fourier transforms with respect to r.)

2. The chain is prepared in a state $|\psi_0\rangle$ which is ground state of $H(m_0)$ and at the time $t = 0$ the mass is abruptly changed to a different value $m \neq m_0$ (this operation is known as *quantum quench*). Consider the time evolution of the operators ϕ_r and π_r and determine the relation between the operators $(A_k, A_k^\dagger)$, corresponding to mass m and dispersion relation Ω_k, and the operators $(A_{0k}, A_{0k}^\dagger)$, corresponding to mass m_0 and dispersion relation Ω_{0k}.

3. Recalling that $A_{0k}|\psi_0\rangle = 0$, calculate the expectation values

$$\begin{aligned} e_1 &= \langle\psi_0|A_{k_1}A_{k_2}|\psi_0\rangle \quad , \quad e_2 = \langle\psi_0|A_{k_1}A_{k_2}^\dagger|\psi_0\rangle \\ e_3 &= \langle\psi_0|A_{k_1}^\dagger A_{k_2}|\psi_0\rangle \quad , \quad e_4 = \langle\psi_0|A_{k_1}^\dagger A_{k_2}^\dagger|\psi_0\rangle. \end{aligned}$$

4. Show that the equal-time two-point correlation function of the Heisenberg operator $\phi_r(t)$ on the state $|\psi_0\rangle$ can be written as

$$\langle\psi_0|\phi_{m+r}(t)\phi_m(t)|\psi_0\rangle \;=\; \sum_{k=0}^{N-1}\frac{\Omega_k^2+\Omega_{0k}^2+(\Omega_k^2-\Omega_{0k}^2)\cos(2\Omega_k t)}{\Omega_k^2\Omega_{0k}}\,e^{2\pi i k r/N}\,.$$