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Fall 2011 - Entrance Examination: Statistical Physics

Solve one of the following problems (one well-solved problem is preferable to many partially addressed ones).

Write down the solution clearly and concisely. State each approximation used. Diagrams and plots are welcome. Number pages, problems and questions clearly.

Do not write your name on any problem sheet, but use the extra envelope.

Problem 1. Driven harmonic oscillator

Consider a quantum particle in one dimension, with mass m and coordinate x. The particle is trapped in the harmonic potential $V(x) = m\omega^2 x^2/2$ and at times $t < 0$ it is in an eigenstate $|n\rangle_0$ of its Hamiltonian H_0. For $t > 0$ a constant external force f is applied to the particle. Indicate by H_f this new Hamiltonian and by $|n\rangle_f$ the corresponding eigenstates.

1. Determine the natural scales X, P and F of the coordinate $x = X\xi$, of the momentum $p = P\pi$, and of the force $f = F\varphi$ which cast $H_f/(2\hbar\omega)$ in the form of a dimensionless Hamiltonian for a particle with dimensionless coordinate ξ and momentum π (with $m = \omega = \hbar = 1$), subject to a force φ. In the rest of the problem express f in terms of φ.

   Determine the spectrum of the Hamiltonians H_0 and H_f and provide a qualitative plot of the latter as a function of φ.

2. Consider the case in which for $t < 0$ the particle is in the ground state $|0\rangle_0$ of the Hamiltonian H_0. Express $|0\rangle_0$ in terms of φ and of the eigenstates $|n\rangle_f$ of H_f. Determine the evolution of the state of the particle for $t > 0$. Show that the probability $p_{0\to n}$ to find the particle in an eigenstate $|n\rangle_f$ of the final Hamiltonian follows a Poisson distribution.

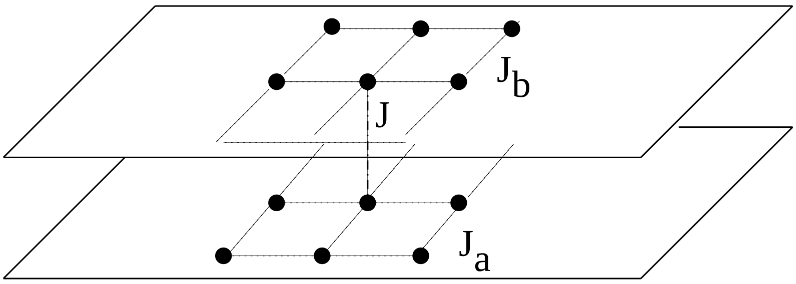
   (*Hint: The ladder operator a_0 for H_0 is conveniently defined as $a_0 \equiv x/X + ip/P$ with $a_0|0\rangle_0 = 0$ and $a_0^\dagger|n\rangle_0 = \sqrt{n+1}|n+1\rangle_0$*)

3. Consider now the case in which for $t < 0$ the particle is in the first excited state $|1\rangle_0$ of the Hamiltonian H_0. Determine the evolution of the state of the particle for $t > 0$ and the probability $p_{1\to n}$ to find the particle in an eigenstate $|n\rangle_f$ of the final Hamiltonian. Provide a qualitative plot of this distribution and discuss its features compared to $p_{0\to n}$. Are there forbidden final states? Under which conditions?

4. On the basis of the result for $p_{1\to n}$ and $p_{0\to n}$, determine the average value $\bar{E}$ of the energy E for $t > 0$. Show that $\bar{E}$ does not depend on φ and explain why. Argue that the variance $\Delta\bar{E}^2$ does indeed depend on φ, and determine this dependence.

Problem 2. Mean field for coupled lattices

Consider a system of Ising spins made of two square lattices A and B, coupled together. Let J_a and J_b be the coupling constants between nearest neighboring spins in the lattices A and B, respectively, and J the coupling constant between the nearest neighboring spins in the two lattices, as indicated in the figure; all the couplings are ferromagnetic.



1. Assume that the coupling between the two square lattices is weak and show that, in the mean field approximation, the spontaneous magnetizations of the two square lattices satisfy the coupled system of equations

$$\begin{aligned} M_a &= f(4J_aM_a + JM_b)\,, \\ M_b &= f(4J_bM_b + JM_a)\,. \end{aligned}$$

   Determine the explicit form of the function f.

2. Determine the critical temperature T_c for which a spontaneous magnetization appears within the system.

3. Determine the magnetic susceptibility for $T > T_c$.

4. Consider now the same system but in the limit $J \to \infty$ of strong coupling between the two square lattices: determine T_c in the mean field approximation.

Problem 3. A random spin system

Consider the Hamiltonian of a random paramagnet

$$E\{\vec{s}\} = -\sum_{i=1}^{N} h_i s_i \tag{1}$$

where $\{\vec{s}\} = (s_1, \ldots, s_N)$, $s_i = \pm 1$ is a spin configuration and h_i are random fields drawn from a Gaussian distribution (density)

$$p(h) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-h^2/2}.$$

1. For a given realization of of $\{h_i\}$ (and finite N), find the ground state configuration $\vec{s}^*$, calculate the entropy S as a function of the temperature T and show that S vanishes as $T \to 0$.

2. Now consider the *annealed* approximation, where instead of the partition function $Z\{\vec{h}\}$ used above — which depends on the specific realization of the random fields — one uses the *annealed* partition function

$$\langle Z \rangle_{\vec{h}} = \int_{-\infty}^{\infty} \mathrm{d}h_1 p(h_1) \ldots \int_{-\infty}^{\infty} \mathrm{d}h_N p(h_N) Z\{\vec{h}\}\,.$$

Calculate the *annealed* entropy (i.e., the entropy in this approximation) and discuss its behavior for $T \to 0$. (*Reminder:* $\int_{-\infty}^{+\infty} \mathrm{d}x\, \mathrm{e}^{-x^2} = \sqrt{\pi}$)

3. For a given configuration $\vec{s}$, let

$$\begin{aligned} \rho(E)\mathrm{d}E &= \mathrm{Prob}\,\{E \leq E\{\vec{s}\} < E + \mathrm{d}E\} \\ &= \int_{-\infty}^{\infty} \mathrm{d}h_1 p(h_1) \ldots \int_{-\infty}^{\infty} \mathrm{d}h_N p(h_N) \delta\left(E + \sum_{i=1}^{N} h_i s_i\right) \mathrm{d}E \end{aligned}$$

be the probability that its energy is in the interval $[E, E + \mathrm{d}E)$.

Calculate the probability density $\rho(E)$.

4. Now consider a different Hamiltonian where the energy E of a spin configuration $\vec{s}$ is a random variable drawn from the probability density $\rho(E)$, independently for each $\vec{s}$. For large N, estimate the number of configurations with energy smaller than a given energy E_0 and from this obtain an estimate of the ground state energy.

(*Reminder:* $\int_{-\infty}^{\epsilon_0} \mathrm{d}x\, \mathrm{e}^{-x^2} \simeq \mathrm{e}^{-\epsilon_0^2}/(-2\epsilon_0)$ *for* $\epsilon_0 \ll -1$)

Consider the *annealed* approximation for this new Hamiltonian and determine the temperature T_c at which the energy $E_{\mathrm{ann}}(T_c)$ calculated within this approximation equals your estimate of the ground state energy. How much is the *annealed* entropy at T_c?

How difficult is it to find the ground state configuration $\vec{s}^*$ of this new Hamiltonian, compared to the case of Eq. (1)?

Problem 4. Expansion of a classical gas

In one space dimension consider a classical thermalized gas at temperature T which is confined at time $t = 0$ by a potential (trap), such that the corresponding state is described by the initial density

$$f(q, p, t = 0) = Q(q)\, F(p),$$

where $F(p) = C\mathrm{e}^{-p^2/(2mk_BT)}$; $Q(q)$ is a (non-negative) with $\int \mathrm{d}q Q(q) = 1$ and C is a normalization constant to be determined from the condition

$$\int \mathrm{d}\Gamma f(q, p, t = 0) = 1,$$

where $\mathrm{d}\Gamma \equiv \mathrm{d}p\, \mathrm{d}q$. The goal of this problem is to study the dynamics of the gas during its *free* expansion at $t > 0$, assuming that the collisions among the particles are negligible.

1. Write down and solve the Liouville equation for $f(q, p, t)$ and comment on the form of the solution.

2. Consider the two particular cases:

   (a) $Q(q) = \delta(q)$;

   (b) $Q(q) = D\mathrm{e}^{-q^2/\sigma^2}$.

   Plot $f(q, p, t)$ in the phase-space plane (q, p) and discuss the obtained findings.

3. Calculate the evolution of the averages $\langle q^2 \rangle$ and $\langle p^2 \rangle$ in both cases (a) and (b): comment the results and discuss how the initial value of the width σ enters the final expression for the computed averages. Compare the results with the corresponding ones for the spreading of a wave-packet in the free one-dimensional Schrödinger equation for a quantum particle.

   (*Reminder:* $\int \mathrm{d}q\, \mathrm{e}^{-q^2} = \sqrt{\pi}$, $\int \mathrm{d}q\, q^2\, \mathrm{e}^{-q^2} = \sqrt{\pi}/2$)

4. Write down the expression of the entropy S in terms of the probability distribution function $f(q, p, t)$ and calculate S as a function of time: comment the result.