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Fall 2012 - Entrance Examination: Statistical Physics

Solve one of the following problems (one well-solved problem is preferable to many partially addressed ones).

Write the solution clearly and concisely. State each approximation used. Diagrams are welcome. Number page, problem and question clearly.

Do not write your name on any problem sheet, but use the extra envelope.

Problem 1. Effect of an electric field on a Aharonov-Bohm quantum ring

Consider an circumference of radius R (in the xy plane) on which there is a particle of charge q and mass M. A magnetic field $\vec{B}$ is applied in the z direction. Let φ be the angle which denotes the position of the particle along the circumference.

1. Write the Schrödinger equation satisfied by the wave function of the particle. *(Hint: Use the minimal substitution $p \to p - \frac{q}{c}A$, where c is the speed of light and A the vector potential.)*
2. Compute eigenfunctions and eigenvalues of the Schrödinger equation.
3. Consider now an electric field along the x direction: what is now the new Hamiltonian? Suppose that the eigenfunctions are written as

$$\Psi(\varphi) = \sum_m C_m e^{im\varphi} : \tag{1}$$

   what is the equation satisfied by the coefficients C_m?

4. What is the qualitative effect of the electric field? Do also a numerical estimate of this effect for $q = -e$, $M = m_e$ (m_e mass of the electron), $R = 20nm$, $B = 1T$ and $E = 10^4 V/m$.

Problem 2. A disordered spin system

Consider a system of N spins $\vec{s} = \{s_1, \ldots, s_N\}$ where each spin takes two values $s_i = \pm 1$. A way to model a disordered system, i.e. a system with impurities and inhomogeneities, is to assume that the energy is a random function of the spin configuration $\vec{s}$.

More precisely, consider the case where the Hamiltonian is defined in the following manner: For each configuration $\vec{s}$ the energy $E(\vec{s})$ is drawn independently at random from a Gaussian distribution with zero mean and variance N:

$$P\{E(\vec{s}) \in [E, E + dE)\} \equiv p(E)dE = \frac{dE}{\sqrt{2\pi N}} e^{-E^2/(2N)} \tag{2}$$

The partition function of the system at temperature T is given by

$$Z = \sum_{s_1=\pm 1} \cdots \sum_{s_N=\pm 1} e^{-E(\vec{s})/T}$$

where we have taken Boltzmann constant $k_B = 1$.

1. Compute the partition function in the *annealed* approximation, namely

$$\langle Z \rangle \equiv \int_{-\infty}^{\infty} \prod_{\vec{s}} dE(\vec{s}) p(E(\vec{s})) Z$$

   and from this compute an approximation for the free energy $F(T)$, the internal energy $U(T)$ and the entropy $S(T)$.

2. Compute the ground state energy E_{GS} assuming that that is the energy where the density of states is equal to one, or equivalently the number of states in the interval $[E_{\text{GS}}, E_{\text{GS}} + dE)$ is of order dE.

3. Compare the value of E_{GS} with the calculation of the internal energy $U(T)$ from point 1 and find the value of the critical temperature T_c below which the annealed approximation breaks down. Discuss the behavior of the entropy around T_c. What is the order of the transition?

4. Consider now a random ferromagnet where the energy is given by

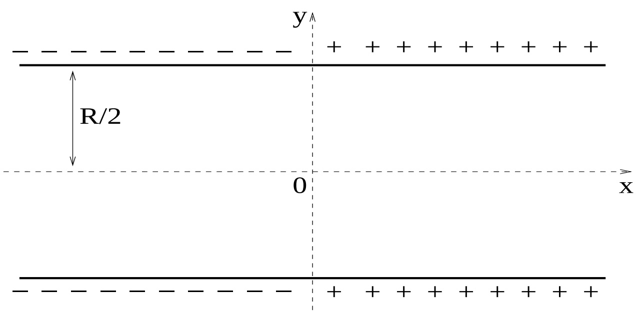
$$E(\vec{s}) = \sum_{i=1}^{N} h_i s_i$$

   where h_i are Gaussian variables drawn independently for each i. Verify that for any $\vec{s}$ the probability that $E(\vec{s}) \in [E, E + dE)$ is exactly given by Eq. (2). Compute the partition function in the annealed approximation (i.e. taking the average of Z on the random fields h_i) and give an estimate of the ground state energy. Show that in this case there is no phase transition. Discuss the difference between the two systems.

Problem 3. Phase separation in two dimensions

Let (x, y) be coordinates on the plane. On the strip $|y| \leq R/2$ consider an Ising model below the critical temperature T_c. If all the spins on the boundaries $y = \pm R/2$ are fixed to the value $+1$, the thermodynamic limit $R \to \infty$ selects the pure phase with positive spontaneous magnetization M; boundary spins all fixed to the value -1 select instead the pure phase with magnetization $-M$.

Consider now and in the following the case in which the boundary spins with $x < 0$ are fixed to the value -1, and those with $x > 0$ are fixed to the value $+1$ (see the figure). As $R \to \infty$ these boundary conditions determine a phase separation that we model in the simplest way: a simple curve (the interface) joining the points $(0, -R/2)$ and $(0, R/2)$ separates the pure phase with magnetization $-M$ on the left from the pure phase with magnetization M on the right. The correlation length is assumed large enough to justify the passage from the lattice to a continuous description.



1. Express the free energy of the system in terms of f_+ and σ, which are the free energy per unit area in the phase with magnetization M and the interfacial free energy per unit length, respectively (call A the total area of the system and l the length of the interface). Denote by $C(L)$ the correlation function of the spins located at the points $(-L, 0)$ and $(L, 0)$; what is the value of $\lim_{L\to\infty} C(L)$?

2. Assume that the interface intersects the axis $y = 0$ only at one point and let $p(u) = \mathcal{N} e^{-2\sigma u^2/R}$ be the probability that the intersection occurs in the interval $(u, u + du)$ on this axis. Determine $\mathcal{N}$ and show that the magnetization $m(x)$ at the point $(x, 0)$ can be written as $m(x) = M\,\mathrm{erf}(\sqrt{\pi}\mathcal{N} x)$, where $\mathrm{erf}(z) \equiv \frac{2}{\sqrt{\pi}} \int_0^z dt\, e^{-t^2}$.

3. Draw a qualitative plot of $m(x)$. Determine $\mathcal{L}(p) \equiv \lim_{R\to\infty} m((\sigma R)^p/\sigma)$ as a function of the parameter p and interpret the result.

4. Suppose now that the system is brought at the critical temperature T_c. Explain what happens and specify σ and $\lim_{R\to\infty} m(x)$.

(Reminder: $\int_{-\infty}^{+\infty} dq\, e^{-q^2} = \sqrt{\pi}$)

Problem 4. Ideal Bose gas in harmonic confinement

Consider a large number N of non-interacting identical bosons at temperature T, confined by the 3D isotropic harmonic potential $V(\mathbf{r}) = \frac{1}{2}m\omega^2 r^2$, being m the mass of the particles and $\mathbf{r}$ the position measured respect to the center of the trapping potential.

1. What is the energy per particle E/N at $T = 0$? Determine the density profile $n(\mathbf{r})$ of this gas at $T = 0$.

2. In the *semi-classical approximation*, one neglects the zero-point energy of the harmonic oscillator and replaces discrete sums over the single-particle eigenstates with continuous integrals of the form $\int_0^\infty d\epsilon g(\epsilon)$, where $g(\epsilon) = \frac{\epsilon^2}{2(\hbar\omega)^3}$ is the density of states at energy ϵ.
Within this approximation, determine the dependence of E as a function of T in the condensed phase, that is where $\mu = 0$ (μ is the chemical potential). **Hint:** use the approximation $\int_0^\infty \frac{x^3}{\exp(x)-1}\mathrm{d}x \cong 6.492$.

3. Below the critical temperature T_C of Bose-Einstein condensation the lowest single-particle eigenstate is macroscopically occupied. Determine T_C as a function of N within the semi-classical approximation, using the condition $N = N_{\mathrm{ex}}(T = T_c, \mu = 0)$, where N_{ex} is the number of particles occupying the exited states. **Hint:** use the approximation $\int_0^\infty \frac{x^2}{\exp(x)-1}\mathrm{d}x \cong 2.404$.

4. Suppose now that our trap contains N linear rigid molecules which have two internal rotational degrees of freedom. The energy levels of such quantum rigid rotors are $\epsilon_l = l(l+1)\hbar^2/(2I)$, with degeneracy $2l + 1$, where $l = 0, 1, 2, \ldots$ and I is the moment of inertia. The molecules do not interact with each other and their size is negligible. If the temperature is in the regime $T_C \ll T \ll \hbar/(2Ik_B)$ (k_B is the Boltzmann constant), how does the specific heat $c = \frac{\partial E}{\partial T}$ depend on T? What is the leading order contribution due to the molecular rotation? **Hint:** the translational degrees of freedom can be treated classically.