Fall 2013 - Entrance examination: Statistical Physics

Solve one of the following problems (one well-solved problem is preferable to many partially addressed ones).

Write down the solution clearly and concisely. State each approximation used. Diagrams and plots are welcome. Number pages, problems and questions clearly.

Important notice:
Do not write your name on any problem sheet, but use the extra envelope.

Problem 1. Spin-spin correlations

Consider a spin model of equilibrium statistical mechanics in dimension $d \geq 2$ which undergoes a second-order phase transition at a critical temperature T_c. Suppose that its continuum limit for $T \geq T_c$ is described by the Euclidean field theory with action

$$\mathscr{A} = \frac{1}{2} \int \mathrm{d}^d x \left(\nabla \phi \cdot \nabla \phi + m^2 \phi^2 \right) , \tag{1}$$

where $x = (x_1, \ldots, x_d)$ denotes a point in d-dimensional Euclidean space, the scalar field $\phi(x)$ is the spin field, and m^2 is proportional to $(T - T_c)$.

1. Determine (up to a normalization constant) the correlation function $G(x) = \langle \phi(x)\phi(0) \rangle$, knowing that the differential equation

$$u'' - \frac{2\alpha}{z} u' - c^2 u = 0$$

   has $u(z) = z^{\alpha+\frac{1}{2}} K_{\alpha+\frac{1}{2}}(cz)$ as the solution which decreases exponentially as $(cz) \to +\infty$, where $K_\nu(x)$ is the modified Bessel function.

2. Determine (up to a normalization constant) $G(x)$ at $T = T_c$. Specify the result for the particular case $d = 2$.

From now on consider $T > T_c$.

3. Determine the correlation length

$$\xi = -\left[\lim_{|x| \to \infty} \frac{\ln G(x)}{|x|} \right]^{-1} , \tag{2}$$

   where $|x|$ is the distance of x from the origin. Determine also the second-moment correlation length

$$\xi_{2nd} = \left[\frac{\int \mathrm{d}^d x \, |x|^2 \, G(x)}{2d \int \mathrm{d}^d x \, G(x)} \right]^{1/2} . \tag{3}$$

4. Relate the following quantity

$$F(x) = \int \mathrm{d}^d y \, \langle \phi^2(y) \phi(x) \phi(0) \rangle$$

   to the correlation function $G(x)$. (*In doing so, set* $\langle \phi^2(y) \rangle = 0$).

Useful formulae:

(A) $\displaystyle\int_0^\infty \mathrm{d}z \, z^\mu K_\nu(az) = 2^{\mu-1} a^{-\mu-1} \Gamma(\frac{1+\mu+\nu}{2}) \Gamma(\frac{1+\mu-\nu}{2})$ for $\mu + 1 \pm \nu > 0, \quad a > 0,$

(B) $\Gamma(x+1) = x\Gamma(x)$ with $\Gamma(1) = 1$.

Problem 2. Interacting Heisenberg spins

Consider three quantum spins $s = 1/2$ interacting with Heisenberg exchange:

$$H[\vec{s}_1, \vec{s}_2, \vec{s}_3] = J_{12}\,\vec{s}_1 \cdot \vec{s}_2 + J_{13}\,\vec{s}_1 \cdot \vec{s}_3 + J_{23}\,\vec{s}_2 \cdot \vec{s}_3, \tag{1}$$

where the spins $\vec{s}_i = (s_i^x, s_i^y, s_i^z)$ are located at some positions $i = 1, 2, 3$ in space.

1. Assume $J_{12} = J_{13} = J_{23} \equiv J$.
   Which is the ground state of H, its energy and degeneracy? Consider both positive and negative values of J. Discuss the various symmetries of $H[\vec{s}_1, \vec{s}_2, \vec{s}_3]$ (in spin and real space) and how they manifest themselves on the spectrum.

2. Now consider $J_{12} = J_{13} \equiv J \neq J_{23}$.
   (a) Discuss again the symmetries of H. Do you expect some of the previous degeneracies to be lifted (just by reasoning, without calculation); why?
   (b) Show that $[H, \vec{s}_2 \cdot \vec{s}_3] = 0$ and provide a physical interpretation of this fact.
   (c) Without explicit calculation, use the symmetries to find the eigenstates of H. Then evaluate their energies and determine the spectrum.

Consider the most general case $J_{12} \neq J_{13} \neq J_{23} \neq J_{12}$.

3. Is the spectrum of H still degenerate? Why? Determine the spectrum and the corresponding eigenstates.

4. Add an Ising interaction to H:

$$H \to H' = H + \sum_{ij} K_{ij} s_i^z s_j^z. \tag{2}$$

   Does this lift the remaining ground-state degeneracy? If not, which term would you add in order to lift this degeneracy completely? Add this term and show within perturbation theory that the degeneracy is indeed lifted.

Problem 3. Phase transition in a classical gas

Consider a gas of particles which can occupy the N sites of a one-dimensional lattice. Let n_i ($i = 1, \ldots, N$) be the number of particles at site i and consider the case $n_i \in \{0, 1\}$ in which at most one particle can be on each site.

The energy $E[\vec{n}]$ of a certain configuration $\vec{n} = (n_1, \ldots, n_N)$ is given by

$$E[\vec{n}] = \max\{i : n_i = 1\}. \tag{1}$$

(Use the convention that $E[\vec{n}] = 0$ if $n_i = 0$ at all sites.) In words, if you draw the lattice in the vertical direction, with site $i = 1$ at the bottom and site $i = N$ at the top, the energy equals the index of the topmost occupied site.

1. Calculate the partition function

$$Z(T) = \sum_{\vec{n}} e^{-E[\vec{n}]/T} \tag{2}$$

[Hint: characterize and calculate the number of configurations with $E[\vec{n}] = k$ and sum over $k = 0, \ldots, N$.]

2. Discuss, for finite N, the high-T and low-T cases of Z and of the associated free energy per site $f = -(T \ln Z)/N$. Consider the thermodynamic limit of f, and show that there is a phase transition at $T = 1/\log 2$. Discuss the properties of this phase transition (is it 1st or 2nd order?). Which other commonly discussed statistical system would you parallel it to?

3. The system discussed above can be considered as a gas of particles in the grand canonical ensemble at chemical potential $\mu = 0$. Calculate the partition function at $\mu \neq 0$.

4. Discuss the behavior of the phase transition as a function of μ. What happens to the critical temperature T when you vary μ away from zero? Would the phase transition occur in a gas with fixed density $\rho = N^{-1} \sum_i n_i$?

Problem 4. Density fluctuations of Fermions

Consider a (non-relativistic) gas of non-interacting Fermions of mass m, in the grand canonical ensemble at temperature T (with $\beta = 1/(k_B T)$) and chemical potential μ. The number N_V of particles contained in a large cubic volume V fluctuates and so does the corresponding spatial density ρ: here we would like to investigate its probability density $p(\rho) \propto \exp\{-(V/\ell^3) I(\rho\ell^3)\}$ for large V, where $\ell = [2m/(\beta\hbar^2)]^{1/2}$ is the thermal wavelength.

1. Assume first that only one single-particle energy level ε is available for the particles within the volume V: determine the probability distribution function $P(n)$ of the (random) occupation number n of this level and express it in terms of $\Lambda \equiv \mathrm{e}^{-\beta\varepsilon+\beta\mu}$.

2. Indicate by ε_k the actual single-particle energy levels of the gas within the volume V and by n_k the corresponding occupation numbers. Which is a convenient choice for the quantum numbers k? Write down the expression of the probability distribution function $P(\{n_k\}_k)$ of the occupation numbers $\{n_k\}_k$. Calculate the *moment generating function* $\langle \mathrm{e}^{-sN_V} \rangle$ of the (random) total number N_V of particles contained in V, with $s \in \mathbb{R}$.

3. The limit $V \to \infty$ can be conveniently studied by introducing the function $\psi(s)$ from $\mathrm{e}^{-V\psi(s)} = \langle \mathrm{e}^{-sN_V} \rangle$. Interpret the values of $\psi(0)$ and $\psi'(0)$. Write down the expression of $\psi(s)$ for $V \to \infty$. For $\mu = 0$ determine the leading asymptotic behaviors of $\psi(s)$ and provide a sketch of $\psi(s)$.

   [Hint: you are not required to calculate ψ explicitly.]

4. Establish a relation between $p(\rho)$ and $\psi(s)$ and note that for $V \to \infty$ the saddle-point method can be used in order to determine $I(x = \rho\ell^3)$. On the basis of the sketch of $\psi(s)$, draw the expected dependence of $I(x)$ on x, highlight its qualitative features, and determine the behavior of $I(x)$ for $x \gg 1$.

   [Hint: $I(0) = \psi(+\infty)$]