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"Le cose fuori del loro stato naturale né vi si adagiano, né vi durano[1]*."*

Gian Battista Vico, Principi D'una Scienza Nuova

[1] Things do not settle or endure out of their natural state.

Introduction

The coherent evolution of a quantum many body system is a topic that has attracted a lot of attention in the last few years (see [3] for a recent review on this topic). The reasons for this interest are several.

First of all, there is an *experimental* motivation. Recent progresses with experimental techniques, in particular in the field of cold atomic gases [4], allow to manipulate quantum systems with an unprecedented degree of accuracy. On the one hand, it is possible to tune (and also change in time) the parameters of these systems. On the other hand, these systems are well isolated from the external environment, therefore it is possible to observe their unitary quantum evolution. This is in contrast with the usual solid state experimental setup, where there is a unavoidable coupling with the environment that introduces dissipation and decoherence.

A second motivation is of *theoretical* nature. Out of equilibrium physics is one of the yet unexplored frontiers of modern physics. While we have a lot of tools to understand equilibrium physics, as well as mean field theory and the renormalization group, non equilibrium physics is not so well understood. Due to the complexity of the problem, it seems an apt strategy to focus our efforts on some simple realizations of out of equilibrium physics. In this thesis, we will study the paradigm of (sudden) *quantum quenches*. In this protocol an extended quantum systems is prepared in a pure state, typically the ground state of its Hamiltonian. Then, we suddenly change the Hamiltonian tuning some parameters and we observe its coherent evolution. While for a finite size system we expect recurrence, it is conceivable that a large system could decay towards a stationary state. The thermodynamical characterization of this (eventual) stationary state is one of the most intriguing puzzles of this field. It has been conjectured that in this regard integrable quantum system could be very special: as they classical counterparts they could not thermalize, i.e. the stationary state of the system cannot be described by a standard thermodynamical ensemble, e.g. the canonical one. Instead, it has been suggested that a proper description of the stationary state for integrable systems is provided by the "generalized Gibbs ensemble", that takes into account all the integrals of motion. These points will be discussed more thoroughly in body of this thesis. Here, we would like only to emphasize that how to describe the stationary state of such systems and the mechanism behind this relaxation are still open problems, also in the non integrable case, even if a lot of nice ideas have been put forward in order to understand these issues. Moreover, this is a problem that can be investigated experimentally: as we have stressed before, it is nowadays possible to experimentally realize systems very close to integrability, to change their parameters in time and to observe their coherent evolution.

Finally, there is also a *technological* motivation for this field of research. It is very plausible that the coherent quantum dynamics will play a major role in future experimental set up and technologies. An example could be provided by a quantum computer, that will definitely require the capability of manipulate interacting system in time. Therefore, a better understanding of out of equilibrium quantum physics could be crucial for the developing of new technologies.

During these last few years, our research work has been focused on the development of two analytical tools to analyze quantum quenches in integrable systems. While it is difficult to overestimate the importance of numerical simulations in order to develop a physical intuition for the out of equilibrium coherent dynamics of extended systems, many of the interesting phenomena we would like to analyze (e.g. the long time behavior of large systems) are quite difficult to grasp with the current numerical techniques. Moreover, as we have discussed, the physics of integrable systems could exhibit an intriguing behavior, qualitatively different from the one of non integrable systems, and still not well understood, that hopefully could be addressed analytically.

This thesis is organized as follows. In chapter 1, we present an introduction to quantum quenches: we discuss some of the most interesting problems in the field as well as some simple solvable model.
In chapter 2, we provide an introduction to integrable quantum field theories, one of the central techniques used in this thesis. Even if we explain the technics used in integrable field theories in a quite detailed way, the emphasis of this chapter is more on the qualitative physical properties that make integrable systems different from the non integrable ones.

Then, in chapter 3, we analyze quantum quenches in integrable field theories, showing that for a certain class of quantum quenches, we can argue that the long time behavior can be described by a generalized Gibbs ensemble. This result is quite interesting, because it shows explicitly that the GGE could play an important role also in interacting integrable systems.
Finally, in chapter 4 we present some progresses on the intriguing idea about the transformations of the Zamolodchikov-Fadeev operators. These operators create and annihilate particles in integrable field theories and satisfy a non trivial algebra involving the S matrix. So, we explore what happen when we change a parameter of the theory and hence the S matrix: can we express the new creation operators in terms of the old ones? This problem, while interesting in itself, has a clear connection with quantum quenches and constitutes a fascinating line of research.

The material presented in this thesis is based on the following papers:

- Fioretto Davide and Mussardo Giuseppe, *Quantum quenches in integrable field theories*, New Journal of Physics. 2010;12:055015
- Sotiriadis Spyros, Fioretto Davide and Mussardo Giuseppe, *On the Initial States of an Integrable Quantum Field Theory after a Quantum Quench*, in preparation.

Chapter 1

An Introduction to Quantum Quenches

In this chapter, we would like to provide a introduction to quantum quenches and thermalization. First of all (sec. 1.1) , we define the problem and explain some of the conjectures that have been put forward to deal with it, both for integrable and non integrable systems. In the same section, we briefly sketch the experimental motivations that spurred the interest towards the coherent dynamics of integrable systems and thermalization. Finally, in sec. 1.2 , we study some exactly solvable models.

1.1 Quantum Quenches and Thermalization

1.1.1 Definition of the Problem

Quantum quenches are one of the simplest possible realization of out of equilibrium physics. Let us consider an extended quantum system in d dimension at zero temperature. Therefore, the system will be in the ground state $|\psi_0\rangle$ of the Hamiltonian H_0. Then, at time $t = 0$, we suddenly change the Hamiltonian from the initial one to a different one, and then we study the coherent dynamics governed by the new Hamiltonian H. We emphasize that, from the theoretical point of view, the initial Hamiltonian H_0 is simply a way to select a proper initial state, whose dynamics is non trivial. The quantum evolution is always dictated by the post-quench Hamiltonian H. Sure enough, there will be some transient effect for this sudden change of the Hamiltonian. However, we are mainly interested in what happens for large times. First of all, does the systems reach a stationary state? Clearly, this is not a possibility for a finite system, where we are bound to have quantum recurrence. However, our

intuition suggests that for large systems (i.e., in the thermodynamic limit), if we focus our attention on a finite portion of the system, the rest will act as a bath and a stationary state will be reached. A natural question is if it is possible to characterize this stationary state in a simple way. It is quite natural to propose that, after a long time, the system will *thermalize*: this means that the stationary state can be described by a thermodynamical ensemble, e.g. the canonical one.
Therefore, we have thermalization if, for *local* operators $\mathcal{O}(x)$, we have

$$\langle\psi_0(t)|\mathcal{O}(x)|\psi_0(t)\rangle \xrightarrow[t\to+\infty]{} \langle\mathcal{O}(x)\rangle_{can} = \frac{\mathrm{Tr}\left[e^{-\beta H}\mathcal{O}(x)\right]}{Z}, \tag{1.1}$$

where $Z = \mathrm{Tr}\left[e^{-\beta H}\right]$, while the effective temperature β is implicitly defined by the constrain that the expectation value of the energy is constant in time, so

$$\overline{E} = \langle\psi_0(t)|H|\psi_0(t)\rangle = \frac{\mathrm{Tr}\left[e^{-\beta H}H\right]}{Z}. \tag{1.2}$$

Alternatively, we could define thermalization in terms of the microcanonical ensemble, investigating if

$$\langle\psi_0(t)|\mathcal{O}(x)|\psi_0(t)\rangle \xrightarrow[t\to+\infty]{} \langle\mathcal{O}(x)\rangle_{mic} = \sum_{\overline{E}\le E_\alpha<\overline{E}+\Delta} \frac{1}{\mathcal{N}}\langle E_\alpha|\mathcal{O}(x)|E_\alpha\rangle, \tag{1.3}$$

where the sum is over the $\mathcal{N}$ eigenstates of the Hamiltonian contained in the microcanonical energy shell $[\overline{E},\overline{E}+\Delta)$. Of course, in the thermodynamic limit these two definition of thermalization are equivalent. We will elaborate more on thermalization in the next subsections. Here, we would like to stress that

- the locality of the operators plays a major role: we cannot expect that (1.1) holds for any operator
- even if it is quite natural, (1.1) is indeed a very strong statement: it means that, from the point of view of local operators, the only thing that the system remembers of its initial condition is its energy.

On the other hand, our experience with classical systems suggests that integrable systems could be very peculiar and do not thermalize. Their thermalization (or lack thereof) is the issue we would like to investigate in this thesis. Before analyzing in a more careful way thermalization in isolated quantum systems, we believe that it could be helpful to briefly consider the experimental situation of the problem.

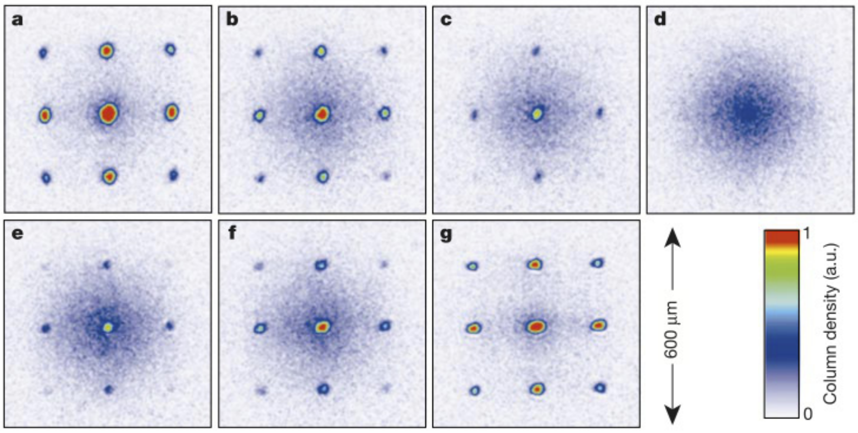

Figure 1.1: Collapse and revival of a matter wave function of a Bose Einstein condensate (from [5])

1.1.2 Experimental Motivations

In this subsection, we would like to briefly present the experimental motivations behind these studies. Since we will not attempt to explain quantitatively the datas of any experiment, in this subsection we do not consider in details any experiment. Instead, we would like to convince the reader, with a couple of examples, that the study of out of equilibrium coherent dynamics in extended systems is a very *timely* problem, and provides a nice interplay between theory and experiments. For many decades, the unitary evolution of an extended system has been seen an academic topic. Indeed, in solid state physics it is usually not possible to decouple a system from it environment, and therefore dissipation and decoherence occur, spoiling the unitary evolution. This point of view is no more legitimate now, since experiments, in particular in cold atomic gases [5–9], have shown that it is possible to minimize the coupling of the system with the environment and therefore observe the coherent evolution of a many body system. This is nicely shown, for example, in figure (1.1), where it is possible to see the revival of a many body wavefunction of a Bose Einstein condensate in an optical lattice.

Moreover, for decades, exactly solvable models have been seen essentially as a low-energy approximation of complex systems. However, the recent technological improvements in the area of cold atomic gases allow the study of systems that are very close to the idealized, exactly solvable ones. These new experimental possibilities make us wonder if it is maybe possible to observe a qualitatively new physics. This possibility is nicely exemplified by the in-

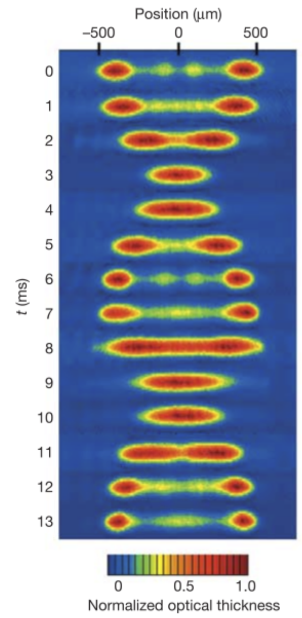

Figure 1.2: Absence of thermalization in a 1 d Bose gas with delta-like interaction, from [6]

fluential work of Kinoshita *et al* [6]. In this work, they studied the out of equilibrium evolution of a one dimensional system with delta-like interaction, placing the atoms in a superposition of $\pm k$ momentum states and observing their evolution. It turns out that that the system does not equilibrate (fig. 1.2) in the experimentally available time scales, a fact that the authors suggested could be linked to the quasi-integrable nature of the system at hand. This experiment spurred a great deal of interest about the coherent dynamics of many body systems: do quantum integrable systems reach thermal equilibrium or not? And, what do we know about non integrable systems? Is their thermalization always guaranteed and well understood?

1.1.3 Thermalization in Non Integrable Systems

After this *excursus* on the experimental motivations behind this field of research, let us go back to the issue of thermalization. For a classical systems, the key concept behind thermalization is *ergodicity*. A system with N degrees of freedom in d spatial dimensions can be represented by a point in the $2dN$ dimensional phase space. Given an initial condition $X_0 = (\boldsymbol{p}_0, \boldsymbol{q}_0)$, the Hamiltonian is ergodic if the trajectory of the system in the phase space covers uniformly the constant energy hypersurface selected by the initial condition, for almost every X_0. This condition allows to replace time averages with phase space averages weighted with the microcanonical ensemble, hence for any operators $\mathcal{O}(\boldsymbol{p},\boldsymbol{q})$ and almost any[1] initial condition X_0, we have

$$\begin{aligned}\langle\mathcal{O}\rangle_{time} &:= \lim_{T\to+\infty}\frac{1}{T}\int_0^T dt\,\mathcal{O}(\boldsymbol{p}(t),\boldsymbol{q}(t)) = \\ = \langle\mathcal{O}\rangle_{mic} &:= \int d^{dN}p\; d^{dN}q\,\mathcal{O}(\boldsymbol{p},\boldsymbol{q})\delta\left[H(\boldsymbol{p},\boldsymbol{q})-H(\boldsymbol{p_0},\boldsymbol{q_0})\right]. \end{aligned} \quad (1.4)$$

However, translating the concept of ergodicity to the quantum domain is a non trivial task. For example, let us consider a non degenerate Hamiltonian with eigenvalues E_α. We select a microcanonical shell $\mathcal{S}=[E_\alpha,E_\alpha+\Delta)$ and we choose as the initial state a superposition of eigenstates in this energy shell, i.e.

$$|\psi_0\rangle=\sum_{\mathcal{S}}c_\alpha|E_\alpha\rangle. \quad (1.5)$$

The time averaged density matrix is expressed by the so-called *diagonal ensemble*

$$\rho_{diag}=\lim_{T\to+\infty}\frac{1}{T}\int_0^T dt|\psi_0(t)\rangle\langle\psi_0(t)|=\sum_{\mathcal{S}}|c_\alpha|^2|E_\alpha\rangle\langle E_\alpha|, \quad (1.6)$$

that coincides with the microcanonical density matrix only for the very special case where all the $|c_\alpha|^2$'s are equal. Therefore, quantum ergodicity in a strict sense is almost never realized. This simple result shows us that we should focus our attention not on the density matrix of the whole system, but on the expectation values of *observables.* A strong and nice result among these lines dates back to Von Neumann [10–12]. The idea is the following: let us consider a set of macroscopic coarse grained observable $\{M_l\}$: these are commuting operators the define a macroscopic state. The idea behind these macroscopic observables is that in quantum mechanics we cannot measure with arbitrarily high precision two observables that do not commute, as the position and momentum. However, the indetermination principle plays no role for macroscopic bodies: therefore, from the mathematical point of view, we could imagine that when we are measuring the position and momentum of a macroscopic body, we are not really measuring the position and momentum quantum operators but coarse-grained commuting operators build from the quantum ones. The statement of Von Neumann's quantum ergodic theorem is that, under suitable assumptions (e.g. the Hamiltonian has no resonances-that means that the energy level differences are non degenerate),

[1] "Almost any" means: a part for a set of measure zero.

for *any* state $|\psi_0\rangle$ in the energy shell $\mathcal{S}$, for *most* of the choices of the set of the macroscopic observables $\{M_\nu\}$ and *most* of the times t, thermalization occurs, i.e.

$$\langle\psi_0(t)|M_l|\psi_0(t)\rangle = \langle M_l\rangle_{mic}. \tag{1.7}$$

Here *most* is used in the technical sense that the appropriate measure of the set of the elements for which the statement of the theorem does not hold is bounded from above and *small* for realistic systems, even if it is not necessarily zero. For a careful statement of the hypothesis of the theorem and of these bounds we refer the reader to the original literature. As a side remark, we notice that the concept of ergodicity that emerges for this theorem is quite different from the classical counterpart, since there is no time average. Therefore, it has been introduced the nomenclature of *normal typicality* to refer to the statement of Von Neumann theorem. While the statement of the theorem is quite general, its application to realistic systems is not so straightforward. Indeed, looking at a concrete many-body system, it is of primary interest not just to find out whether in principle a set of macroscopic observables that behave ergodically exists, but whether specific and natural observables, such as the magnetization for spin chains, density for cold atomic gases, or various correlation functions thermalize or not.

Another intriguing way to understand thermalization is the *eigenstate thermalization hypothesis* [13–15] . The objects of this hypothesis are the expectation values of observables on the basis of the eigenstates of the Hamiltonian $\langle E_\alpha|O|E_\alpha\rangle$, and the statement is that for *natural, physically interesting* observables (whatever it means) these expectation values are a smooth and quite flat function of the energy, with the possible exception of the extremes of the spectrum. Therefore, if the initial state is peaked around an energy (i.e. the $|c_\alpha|^2$ are peaked around an energy $\hat{E}$, and outside a neighborhood of $\hat{E}$ they are essentially zero), the eigenstate thermalization hypothesis implies the equivalence of the diagonal (1.6) and the microcanonical ensemble. Recently, the connection between this hypothesis and Von Neumann's theorem have been emphasized in [16].

So, summarizing

1. Even if quantum thermalization is quite a natural concept, there is still no conclusive evidence of the physical mechanism behind it, and what are the most general hypotheses under which it occurs.

2. Thermalization cannot hold in general, but only for a class of states (e.g. states that are sufficiently peaked in the energy space) and observables.

3. Even if thermalization has been observed in many numerical simulations, some violations of it are also known. A very clear one was pointed

out in a simulation of an infinite long Ising spin chain with parallel and transverse field [17], whose Hamiltonian is

$$H = -\sum_i \sigma_i^z \sigma_{i+1}^z - h \sum_i \sigma_i^z - g \sum_i \sigma_i^x \tag{1.8}$$

The authors of [17] studied the long times behavior of the reduced density matrix, up to three sites. If this reduced density matrix tends to a thermal one, then for any local observable (i.e. for any observable that acts on up to three sites) thermalization occurs; otherwise, it means that there exist local observables that do not thermalize. It turns out that if the initial state is $|y+\rangle$, i.e. all the spins are aligned in the positive y direction, then the reduced density matrix tends to the canonical one, while if the initial state is $|z+\rangle$ there is no strict convergence for $t \to +\infty$ since oscillations persist even at large times. However, if we take a time average (as in the classical case), we get rid of these oscillations in time and we have a convergence to the thermal distribution. More interestingly, if the initial state is $|x+\rangle$, we see convergence to a density matrix that is different from the thermal one and therefore thermalization does not occur. Of course, since these results are numerical, it is not possible to reach infinite long times and it may be that actually thermalization does occur but at a time scale beyond the ones reached by the simulation. However, we believe that this result is very interesting, in particular because these different behaviors are exemplified by very simple initial states.

1.1.4 Integrability and Thermalization

Let us now consider integrable systems. In the classical case, integrable systems are a very well known example of non ergodic behavior: they possess as many integrals of motion as the number of the degrees of freedom and therefore they cannot explore the full hypersurface of constant energy. The experiment of Kinoshita *et al.* [6] drew the attention of the community to the quantum case: does quantum integrability prevent thermalization? An interesting conjecture on this regard was made by Rigol *et al.* [18] . Their idea, supported by numerical simulations for a lattice hard core boson gas in one dimension, was that the long times behavior of integrable systems is not described by a canonical density matrix, but by a generalized Gibbs ensemble that takes into account the extra integral of motion I_l, i.e.

$$\hat{\rho}_{gen} \sim e^{-\sum_l \alpha_l I_l}, \tag{1.9}$$

where α_l are Lagrange multipliers fixed by the the initial state $|\psi_0\rangle$ through the condition

$$\langle\psi_0|I_l|\psi_0\rangle = Tr\left[\hat{\rho}_{gen} I_l\right]. \tag{1.10}$$

This conjecture is indeed quite appealing, since it reminds the classical idea that the dynamics of integrable models is constrained by their conserved quantities. However, it turns out that quantum integrability is quite a subtle concept, and its relation with the existence of conserved quantities is not so simple as in the classical case. We will discuss extensively the difficulties in the definition of quantum integrability in sec. 2.2. For example, one point is that we cannot ask for functional independence of commuting operators-they are always functional dependent. So, the best we could achieve is algebraic independence, but a set of algebraic independent commuting conserved quantities is constituted by the powers of the Hamiltonian, or the projectors over the energy eigenstates: should we conclude that all the Hamiltonian are integrable?And, in the spirit of this conjecture, should we include the powers of the Hamiltonian in the canonical ensemble? However, we should keep in mind that in statistical mechanics we demand that two large (macroscopic) portion of the same system are statistically independent, i.e. the state of one subsystem does not affect the probabilities of various states of the other subsystems. This implies that the only conserved quantities that can enter in (1.9) are the ones that are local[2] hence additive. This explain why in the canonical ensemble H^2 doesn't appear, and suggests that only local conserved quantities should enter in (1.9).

At the same time, the relation between quantum integrability and conserved quantities is not so clear, as in the classical case. We will discuss this difficulties in section 2.2. Here we would like to stress that the best available definition of integrability refer to the absence of diffraction in scattering and, at least in the continuum, it is probably connected to the existence of a suitable number of local conserved quantities: as Sutherland writes [19], "*Nondiffraction could probably be insured by the existence of a complete set (as many operators as the number of particles) of independent local operators that commute with the Hamiltonian H, and with each other*". This idea is essentially the one used in the early arguments for the integrability of relativistic field theories (see sec. 2.3), and the word independent should be understood as follows. We have a set of conserved quantities Q_s that are diagonal on one-particle plane waves:

$$Q_s|p\rangle = \omega_s(p)|p\rangle. \tag{1.11}$$

[2]More precisely, we should say that they are integrals over the whole space of local a density. However, we will sometimes call these conserved quantities local,with a slight abuse of language.

In this contest, independence means that the one particles eigenvalues $\omega_s(p)$ are independent function of p [20]. Therefore, Rigol *et al.*'s conjecture is at least well posed and physically sound. Its validity has been the subject of several studies, mostly done by using specific models [21–28], and also investigated for initial thermal distributions [29]. An important step forward was taken in the work of Barthel and Schollwöck [30] in which they generalized a previous result by Calabrese and Cardy [31], proving rigourosly that for Gaussian initial states and quadratic (fermionic or bosonic) systems the conjecture does hold: a finite subsection of an infinite system indeed relaxes to a steady state described by a generalized Gibbs ensemble where the extra integrals of motions are simply the occupation numbers of each eigenmode[3]. Moreover, they stated (postponing the proof to a future paper) that a similar result holds also for Bethe-ansatz solvable models, even if in this case the extra integrals of motions appearing in the density matrix do not have such a simple physical interpretation.
We will come back to the generalized Gibbs ensemble in chapter 3 , where we will analyze the thermalization of integrable field theories.

1.2 Quantum Quenches in Solvable Models

In this section, we would like to analyze quantum quenches in some solvable model. In this way, we introduce some of the ideas that will play a major role in the next chapters.

1.2.1 Conformal Field Theories

Quantum quenches could be seen as the analytical continuation at real times of a statistical physics problem in a strip. In order to illustrate this point, let us consider the expectation value of an operator $\mathcal{O}$ at time t on a state $|\psi_0\rangle$, i.e.

$$\langle \mathcal{O}(t)\rangle = \langle \psi_0|\mathcal{O}(t)|\psi_0\rangle = \langle \psi_0|e^{iHt}\mathcal{O}e^{-iHt}|\psi_0\rangle. \tag{1.12}$$

First of all, let us introduce a parameter $\tau_0 > 0$, that is useful to regularize the divergencies that may arise in the previous expression. So, $|\psi_0\rangle \to e^{-H\tau_0}|\psi_0\rangle$. Of course, τ_0 could be an useful tool in the intermediate steps of the calculations, but at the end of the day we would like to set $\tau_0 = 0$ or at least give a physical meaning to τ_0. So, if we do an analytical continuation $it = \tau$, we

[3] As a matter of fact, even if local conserved quantities should play a major role, as we have emphasized in the text, it is often simpler to derive the generalized Gibbs ensemble in terms of occupation numbers.

have that

$$\langle \mathcal{O}(t) \rangle \rightarrow \frac{\langle \psi_0 | e^{-H(\tau_0 - \tau)} \mathcal{O} e^{-H(\tau_0 + \tau)} | \psi_0 \rangle}{\langle \psi_0 | e^{-2H\tau_0} | \psi_0 \rangle}. \tag{1.13}$$

So, we have mapped the original out of equilibrium quantum problem in $d+1$ dimension in a statistical physics equilibrium problem in a slab in $d+1$ spatial dimensions: the initial state $|\psi_0\rangle$ plays the role of a boundary condition, while $e^{-H\tau}$ is the transfer matrix in the imaginary time direction. Incidentally, $2\tau_0$ is the width of the slab, and this explain why we introduced a regulator τ_0. So, this mapping suggests that we could translate the results of statistical physics in a confined geometry into quantum quench ones. Even if for sake of simplicity we have considered only one operator $\mathcal{O}(t)$, this mapping holds also for products of operators at different times $\mathcal{O}(t_1) \dots \mathcal{O}(t_n)$.
This mapping is particularly useful if the Hamiltonian H is at a critical point of a 1+1 quantum system [31,32]. Indeed, this means that the corresponding bulk Hamiltonian in the strip will flow, under the renormalization group, to a conformal field theory. Moreover, any translational invariant boundary conditions will flow to one of the conformal invariant boundary conditions [33]. Therefore, we can exploit the conformal invariance of the theory and of the boundary to compute correlation functions. However, we should emphasize that, if we put $\tau_0 = 0$ at the end of the calculations, we obtain divergencies. This is a well know phenomenon in the physics of boundary critical phenomena: the correlators at the critical point do not depend only on the universal property of the theory (bulk conformal field theory and conformal invariant boundary state). Instead, they depend also on a non universal length scale (in our case time scale) τ_0, known as the extrapolation length, that is determined by the original boundary conditions. In the quench problem, τ_0 can be associated to the correlations of the initial state.
Conformal invariance simplifies a lot the computation of the correlators of primal operators, that are the operators that, under a conformal mapping $w(z)$ transform as

$$\varphi(w) = w'(z)^{-x} \varphi(z), \tag{1.14}$$

where x is the scaling dimension of the operator. Under the conformal transformation $w = \frac{2\tau_0}{\pi} \mathrm{Log}(z)$, the strip of width $2\tau_0$ is mapped conformally into the upper half plane (fig.1.3). The one point correlation function in the upper half plane is known and quite simple. Indeed, conformal invariance implies that

$$\langle \varphi(z) \rangle_{UHP} = A^{\varphi}_{\psi_0} \left[2Im(z) \right]^{-x}, \tag{1.15}$$

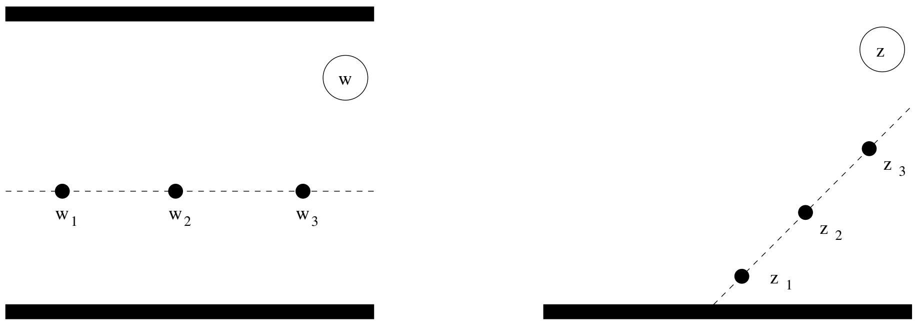

Figure 1.3: The conformal mapping between the strip and the upper half plane.

where $A^{\varphi}_{\psi_0}$ is a (known, at least for diagonal minimal models) constant, that depends on the operator and the boundary condition. Therefore, we have

$$\langle\varphi(t)\rangle = A^{\varphi}_{\psi_0}\left[\frac{4\tau_0}{\pi}\cosh\left[\frac{\pi t}{2\tau_0}\right]\right]^{-x} \underset{t\to+\infty}{\sim} A^{\varphi}_{\psi_0}\left[\frac{\pi}{2\tau_0}\right]^{x} e^{-\frac{\pi x\, t}{2\tau_0}}. \tag{1.16}$$

So, we see that the expectation value of primary operators has a well defined long time limit, and the approach to this value is exponential in time. The decay rate of this approach is non universal: this is not a surprise, since it is not dimensionless. However, the ratio of two decay rates is universal, and it is given by the ratio of the scaling dimensions.
The two point function is a little more complicated, since conformal invariance do not fix univoquely its structure. Indeed, we have

$$\langle\varphi(z_1)\varphi(z_2)\rangle = \left[\frac{z_{1\overline{2}}z_{2\overline{1}}}{z_{12}z_{\overline{12}}z_{1\overline{1}}z_{2\overline{2}}}\right]^{x} F(\eta), \tag{1.17}$$

where $z_{12} = |z_1 - z_2|$ and $z_{1\overline{2}} = |z_1 - \overline{z}_2|$ (the overline denotes the complex conjugate), while F is an arbitrary function of the ratio

$$\eta = \frac{z_{1\overline{1}}z_{2\overline{2}}}{z_{1\overline{2}}z_{\overline{1}2}}. \tag{1.18}$$

Even if the conformal symmetry does not fix the form of the function F, in order to study $\langle\varphi(x_1,t)\varphi(x_2,t)\rangle$ at large distances $r = |x_1 - x_2|$ and long times t, (respect to τ_0) it is sufficient to know the behavior of $F(\eta)$ for small η (close to the surface) and for $\eta = 1$ (deep in the bulk). When $\eta = 1$, we should obtain the two point function in the bulk, hence $F(1) = 1$. Instead, for small η, we have

$$F(\eta) \sim \left(A^{\varphi}_{\psi_0}\right)^2 \eta^{x_b}, \tag{1.19}$$

where x_b is the boundary scaling dimension of the leading boundary operator to which φ couples. Notice that if $\langle\varphi\rangle \neq 0$, $x_b = 0$. So, we have

$$\langle\varphi(r,t)\varphi(0,t)\rangle \sim \begin{cases} [\langle\varphi(t)\rangle]^2 & \text{if } \langle\varphi(t)\rangle \neq 0,\ \tau_0 \ll t < \frac{r}{2} \\ e^{-\frac{\pi x\, r}{\tau_0}} & \text{if } \tau_0 \ll \frac{r}{2} < t \end{cases} . \tag{1.20}$$

So, if $\langle\varphi\rangle \neq 0$, we have that the first term in the expansion of the connected correlator is simply zero for $t < \frac{r}{2}$. This is know as the horizon effect, and it can be understood in the following way. The boundary state is an highly excited state (respect to the ground state) and therefore it is a source of quasiparticles. For $\tau_0 \approx 0$, quasiparticles created at two different points are essentially uncorrelated: therefore, correlations are mediated only by quasiparticles created at the same point. But in a conformal field theory these quasiparticles must travel at the speed of light (that is 1 in our units). Therefore, no correlation is possible for $t < \frac{r}{2}$.
Instead, for $t > \frac{r}{2}$ the (leading term of the) two point function is time independent: this shows that the correlations saturate immediately at $t = \frac{r}{2}$. Notice that the decay is exponential and not power law.

1.2.2 Free Many Body Systems

In this subsection, we would like to analyze free systems. We will focus our attention not only on the physical properties of these systems (i.e. the behavior of correlators), but also on the form of the initial state for the simplest kind of quenches (a change of the frequencies). Indeed, this form will inspire us some of the ideas of chapters 3 and 4.

First of all, let us consider a single particle quantum harmonic oscillator, whose Hamiltonian is

$$H_0 = \frac{p^2}{2\,m_0} + \frac{1}{2} m_0\,\omega_0^2 x^2. \tag{1.21}$$

It is well know that the eigenstates of this problem can be constructed by introducing the ladder operators a_0 , $a_0^\dagger$

$$\begin{pmatrix} x \\ p \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \lambda_0{}^{-\frac{1}{2}} & \lambda_0{}^{-\frac{1}{2}} \\ -i\,\lambda_0{}^{\frac{1}{2}} & i\,\lambda_0{}^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} a_0 \\ a_0^\dagger \end{pmatrix} = V(\lambda_0) \begin{pmatrix} a_0 \\ a_0^\dagger \end{pmatrix}, \tag{1.22}$$

where λ_0 is the a dimensional parameter $\lambda_0 = m_0\,\omega_0$. We assume that, at time $t = 0^-$, the system is prepared in the ground state $|\psi_0\rangle$ univocally determined (up to a phase factor) by the equation

$$a_0\,|\psi_0\rangle = 0. \tag{1.23}$$

Then,at time $t = 0^+$,we quench our Hamiltonian by suddenly varying the parameters $(m_0\,,\omega_0) \to (m\,,\omega)$. Since the position and the momentum operator are continuous at $t = 0$,we have

$$\left(\begin{array}{c} x \\ p \end{array}\right) = V(\lambda_0)\left(\begin{array}{c} a_0 \\ a_0^\dagger \end{array}\right) = V(\lambda)\left(\begin{array}{c} a \\ a^\dagger \end{array}\right), \tag{1.24}$$

hence we can determine the relation between the old ladder operators a_0 , $a_0^\dagger$ and the new ones a , $a^\dagger$

$$a = ca_0 + da_0^\dagger, \qquad a^\dagger = ca_0^\dagger + da_0 \tag{1.25}$$

$$a_0 = ca - da^\dagger, \qquad a_0^\dagger = ca^\dagger - da, \tag{1.26}$$

where

$$c = \frac{\lambda + \lambda_0}{2\sqrt{\lambda\lambda_0}}, \qquad d = \frac{\lambda - \lambda_0}{2\sqrt{\lambda\lambda_0}}. \tag{1.27}$$

We see that this is simply a Bogoliubov transformation and the condition

$$c^2 - d^2 = 1 \tag{1.28}$$

is a direct consequence of the algebra satisfied by the ladder operators

$$[a_0, a_0^\dagger] = [a, a^\dagger] = 1. \tag{1.29}$$

The transformation (1.25) provides us with a simple way to calculate the evolution of any operator $\mathcal{O}(t)$ that can be expresses as a polynomial of $a(t), a^\dagger(t)$. First of all, the time evolution of the ladder operators is trivial, since $a(t) = e^{-i\omega t}a$. Therefore, the only non trivial step in order to compute the expectation values of $\mathcal{O}(t)$ is the calculation of the expectation values of a, $a^\dagger$ (and their products) over the initial state $|\psi_0\rangle$, but this can be easily accomplished using (1.25).

Otherwise, we could try to express the initial state $|\psi_0\rangle$ in terms of the basis of the new Hamiltonian, constructed by acting with the $a^\dagger$ operator on the new ground state $|0\rangle$ ($a|0\rangle = 0$). Since $|\psi_0\rangle$ satisfies the equation

$$(ca - da^\dagger)|\psi_0\rangle = 0, \tag{1.30}$$

we have

$$|\psi_0\rangle = \mathcal{N} \exp\left[\frac{\lambda - \lambda_0}{\lambda + \lambda_0}\, {a^\dagger}^2\right]|0\rangle, \tag{1.31}$$

as it can be proved by using the identity $[a, f(a^\dagger)] = f'(a^\dagger)$. This is a general property of the Bogoliubov transformation: the old ground state written in the new basis has always the form (1.31), the so-called *squeezed vacuum*. These arguments can be straightforwardly generalized to free bosonic or fermionic systems.

Free Bosonic System. Following [31], we analyze a chain of N coupled harmonic oscillator

$$H_0 = \frac{1}{2}\sum_{n=0}^{N-1}\left[\pi_n^2 + (m^0)^2\,\varphi_n^2 + \sum_{j=0}^{N-1}(\omega_j^0)^2\,(\varphi_{n+j}-\varphi_n)^2\right]. \tag{1.32}$$

This system can be diagonalized in momentum space. If we introduce the Fourier transform $\varphi_k = \frac{1}{\sqrt{N}}\sum_n \exp(\frac{2\pi i\,k\,n}{N})\varphi_n$, we have

$$H = \sum_{k=0}^{N-1}\left[\Omega_k^0 A_k^{0\dagger}A_k^0 + \frac{1}{2}\right], \tag{1.33}$$

$$(\Omega_k^0)^2 = (m^0)^2 + 2\sum_{j=0}^{N-1}(\omega_j^0)^2\left[1-\cos\left(\frac{2\,\pi\,k\,j}{N}\right)\right], \tag{1.34}$$

$$A_k^0 = \frac{1}{\sqrt{2\Omega_k^0}}\left(\Omega_k^0\varphi_k + i\pi_k\right), \tag{1.35}$$

$$A_k^{0\dagger} = \frac{1}{\sqrt{2\Omega_k^0}}\left(\Omega_k^0\varphi_{-k} - i\pi_{-k}\right), \tag{1.36}$$

$$\tag{1.37}$$

and the ground state $|\psi_0\rangle$ is characterized by the equation

$$A_k^0|\psi_0\rangle = 0. \tag{1.38}$$

Let us prepare the system in its ground state and quench the frequency $\Omega_k^0 \to \Omega_k$. The relation between the pre-quench ladder operators $A_k^0, A_k^{0\dagger}$ and the the post-quench ones $A_k^0, A_k^{0\dagger}$ is a Bogoliubov transformation

$$\begin{aligned} A_k &= c_k A_k^0 + d_k A_{-k}^{0\dagger}, \qquad & A_k^\dagger &= c_k A_k^{0\dagger} + d_k A_{-k}^0,\\ A_k^0 &= c_k A_k - d_k A_{-k}^\dagger, \qquad & A_k^{0\dagger} &= c_k A_k^\dagger - d_k A_{-k}, \end{aligned} \tag{1.39}$$

$$c_k = \frac{\Omega_k + \Omega_k^0}{2\sqrt{\Omega_k\Omega_k^0}}, \qquad d_k = \frac{\Omega_k - \Omega_k^0}{2\sqrt{\Omega_k\Omega_k^0}}. \tag{1.40}$$

Therefore,the state $|\psi_0\rangle$ is a coherent state

$$|\psi_0\rangle = \mathcal{N}\exp\left[\sum_{k=0}^{N-1} K_{boson}(k) A_k^\dagger A_{-k}^\dagger\right]|0\rangle, \tag{1.41}$$

where

$$K_{boson}(k) \;=\; \frac{\Omega_k^0 - \Omega_k}{\Omega_k^0 + \Omega_k} \tag{1.42}$$

This quantity can be written in a suggestive way going to the continuum limit, where $\Omega(k) = \sqrt{k^2+m^2}$, $\Omega^0(k) = \sqrt{k^2+m_0^2}$ and the process can be interpreted as a quench of the mass of the particle excitation. Let us now introduce the rapidities of the particle relative to the initial and final situations, i.e.

$$\begin{aligned} \Omega^0 = m_0\cosh\xi \quad &, \qquad k = m_0\sinh\xi \\ \Omega = m\cosh\theta \quad &, \qquad k = m\sinh\theta \end{aligned} \tag{1.43}$$

From the equality of the initial and final momenta, we have the relation which links the two rapidities

$$m_0\sinh\xi = m\sinh\theta \quad \Rightarrow \frac{m_0}{m} = \frac{\sinh\theta}{\sinh\xi} \tag{1.44}$$

Therefore, the amplitude $K_{boson}(k)$ of eq. (1.42) can be neatly written as

$$\begin{aligned} K_{boson}(\theta,\xi) \;&=\; \frac{m_0\cosh\xi - m\cosh\theta}{m_0\cosh\xi + m\cosh\theta} = \frac{\frac{m_0}{m}\cosh\xi - \cosh\theta}{\frac{m_0}{m}\cosh\xi + \cosh\theta} = \\ &=\; \frac{\sinh\theta\cosh\xi - \sinh\xi\cosh\theta}{\sinh\theta\cosh\xi + \sinh\xi\cosh\theta} = \frac{\sinh(\theta-\xi)}{\sinh(\theta+\xi)} \;. \end{aligned} \tag{1.45}$$

Free Fermionic System. One can easily work out the Bogoliubov transformation relative to the quench of the mass of a free fermionic system [34]. Consider, in particular, a free Majorana fermion in (1+1) dimension, with the mode expansion of the two components of this field given by

$$\begin{aligned} \psi_1(x,t) \;&=\; \int_{-\infty}^{+\infty} dp\,\left[\alpha(p)A(p)e^{-iEt+ipx} + \overline{\alpha}(p)A^\dagger(p)e^{iEt-ipx}\right] \\ \psi_2(x,t) \;&=\; \int_{-\infty}^{+\infty} dp\,\left[\beta(p)A(p)e^{-iEt+ipx} + \overline{\beta}(p)A^\dagger(p)e^{iEt-ipx}\right] \end{aligned} \tag{1.46}$$

where

$$\begin{aligned} \alpha(p) = \frac{\omega}{2\pi\sqrt{2}}\frac{\sqrt{E+p}}{E} \quad &, \qquad \overline{\alpha}(p) = \frac{\overline{\omega}}{2\pi\sqrt{2}}\frac{\sqrt{E+p}}{E} \\ \beta(p) = \frac{\overline{\omega}}{2\pi\sqrt{2}}\frac{\sqrt{E-p}}{E} \quad &, \qquad \overline{\beta}(p) = \frac{\omega}{2\pi\sqrt{2}}\frac{\sqrt{E-p}}{E} \end{aligned} \tag{1.47}$$

with $\omega = \exp(i\pi/4)$. At $t = 0$, i.e. at the instant of the quench, we can extract the Fourier mode of each component of the Majorana field

$$\psi_i(x,0) \;=\; \int dp\, \hat{\psi}_i(p) e^{ipx} \;\;, \tag{1.48}$$

given by

$$\begin{aligned} \hat{\psi}_1(p) &= \alpha(p)A(p) + \overline{\alpha}(-p)A^\dagger(-p) \\ \hat{\psi}_2(p) &= \beta(p)A(p) + \overline{\beta}(-p)A^\dagger(-p) \end{aligned} \tag{1.49}$$

Suppose now that the mass of the field is changed from m_0 to m at $t = 0$ and let's denote by $(A_0(p), A^\dagger(p))$ and $(A(p), A^\dagger(p))$ the sets of oscillators before and after the quench. The proper boundary condition associated to such a situation is the continuity of the field components before and after the quench

$$\psi_i^0(x, t=0) \;=\; \psi_i(x, t=0) \;\;, \tag{1.50}$$

which implies

$$\hat{\psi}_i^0(p) \;=\; \hat{\psi}_i(p) \;\;. \tag{1.51}$$

This gives rise to the Bogoliubov transformation between the two sets of oscillators

$$\begin{aligned} A_0(p) &= u(p)A(p) + iv(p)A^\dagger(-p) \\ A_0^\dagger(p) &= u(p)A^\dagger(p) - iv(p)A(-p) \end{aligned} \tag{1.52}$$

where

$$\begin{aligned} u(p) &= \frac{1}{2E}\left[\sqrt{(E_0+p)(E+p)} + \sqrt{(E_0-p)(E-p)}\right] \\ v(p) &= \frac{1}{2E}\left[\sqrt{(E_0-p)(E+p)} - \sqrt{(E_0+p)(E-p)}\right] \end{aligned} \tag{1.53}$$

Notice that these functions satisfy the relations $u(p) = u(-p)$ and $v(p) = -v(-p)$ together with $u^2(p) + v^2(p) = E/E_0$, which refers to the normalization of the respective set of oscillators.

The boundary state corresponding to this quench can be written as

$$|\, B\rangle = \exp\left[\int_{-\infty}^{\infty} dp\; K_{fermion}(p) A^\dagger(p) A^\dagger(-p)\right] |\, 0\rangle \;\;, \tag{1.54}$$

where

$$K_{fermion}(p) \;=\; -K_{fermion}(-p) \;=\; i\frac{\sqrt{(E_0-p)(E+p)} - \sqrt{(E_0+p)(E-p)}}{\sqrt{(E_0+p)(E+p)} - \sqrt{(E_0-p)(E-p)}} \tag{1.55}$$

As in the bosonic case, this quantity can be expressed in a more net form by introducing the rapidities of the particle before and after the quench, i..e

$$E_0 \pm p = m_0 e^{\pm\xi} \quad , \quad E \pm p = m e^{\pm\theta} \ . \tag{1.56}$$

Substituting these expressions in (1.55), we get

$$K_{fermion}(\theta, \xi) \ = \ i \, \frac{\sinh\left(\frac{\theta - \xi}{2}\right)}{\sinh\left(\frac{\theta + \xi}{2}\right)} \ . \tag{1.57}$$

Chapter 2

Integrable Field Theories and Boundary Conditions

In this chapter, we would like to review the main properties of integrable field theories in 1+1 dimensions. After a brief exposition of the basic concept in quantum theory (sec. 2.1), in sec. 2.2 we critically consider the different proposed definitions of quantum integrability. In general, quantum integrability is not a consequence of the existence of a suitable number of conserved quantities. However, in sec. 2.1 we show that for relativistic field theory in 1+1 dimension this is the case: quantum integrability is implied by the existence of (countable) infinite many *local* conserved quantities. In the sec. 2.4, we will discuss how such a theory can be solved (i.e. a proper quasi particle basis can be constructed) studying the analytical properties of the S matrix in the complex plane. In the same section we will show that it is also possible to compute the matrix elements of local operators in the quasiparticle basis through the form factors program. Finally, in sec. 2.5 we discuss integrable field theories in presence of a boundary, a physical situation that has a clear link with quantum quenches.

The aim of this chapter is to provide a self consistent introduction to the general and universal properties of integrable field theories. After this exposition, the reader should be able to appreciate both the motivations of our work both the quite sophisticated techniques involved. Therefore, we will not focus our attention on any specific theory, and we will not discuss peculiar (or pathological) properties of a given theory, since they play no role in our subsequent analysis. For these topics, we refer the interested reader to the literature cited in the text.

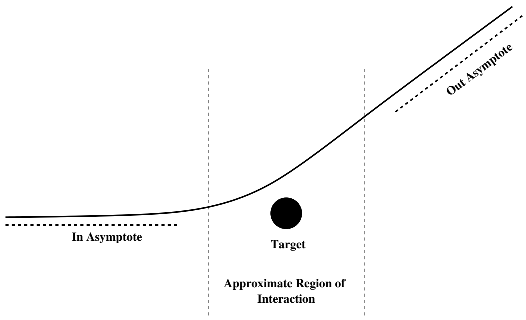

Figure 2.1: The scattering of a classical particle off a fixed target.

2.1 Basic Concepts of Quantum Scattering Theory

In this section we would like to review the basic concepts of quantum scattering theory, setting up the notation for the next sections. Our exposition will be necessarily quite brief: a more thoughtful analysis can be found in [35]. Since the emphasis of this section is on the general ideas rather than on the computational tools, we will focus our attention on the simplest example of scattering: the one of a probe particle (of mass m) off a fixed target. If the interaction between the probe and the scatterer is sufficiently short-ranged, we can safely assume that the classical trajectory of the particle will be a straight line far away from the target, and it will be significantly different from a free trajectory only in a small region near the scatterer (see fig. 2.1). In formulas, if $\boldsymbol{x}(t)$ is the trajectory of the particle, we have:

$$\boldsymbol{x}(t) \sim \boldsymbol{q}_{in} + t\,\boldsymbol{v}_{in} \qquad t \to -\infty\,, \tag{2.1}$$

$$\boldsymbol{x}(t) \sim \boldsymbol{q}_{out} + t\,\boldsymbol{v}_{out} \qquad t \to +\infty\,. \tag{2.2}$$

The scattering trajectories just described are not the only possible orbits of the systems. In general there will be also some bounded orbits, where the particle never escape from the potential of the target. The scattering orbits and the bounded ones make up all the possible orbits of the system. As a matter of fact, in many experimental set up we are not able to follow the trajectory of the particle in the interacting region.This is particularly true when we approach the quantum physics, where the interacting region could be of the size of few atomic diameters. Therefore, it is quite natural to look for a theory of scattering that focus its attention on the asymptotic properties of the orbit far from the scatterer (the *asymptotic region*) rather than on the precise details of the trajectory near the target.

Let us analyze the quantum case. The probe is described by a wave function $|\psi(t)\rangle$ obeying the(one particle) Schrödinger equation

$$i\frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle, \tag{2.3}$$

where H is the sum of the free Hamiltonian $H_0 = \frac{p^2}{2m}$ and the interaction potential V. Let us assume that at time $t=0$ the wave function is $|\psi\rangle$: this condition will fix univoquely the orbit. The asymptotic condition now reads as:

$$U(t)|\psi\rangle \sim U_0(t)|\psi\rangle_{in}, \qquad t \to -\infty, \tag{2.4}$$

for some in state $|\psi\rangle_{in}$, and

$$U(t)|\psi\rangle \sim U_0(t)|\psi\rangle_{out}, \qquad t \to +\infty, \tag{2.5}$$

for some vector $|\psi\rangle_{out}$. Here $U(t)$ is the evolution operator of the full Hamiltonian H, $U(t) = \exp\left[-i\,t\,H\right]$, while $U_0(t)$ is the evolution operator associated to the free Hamiltonian H_0 , $U_0(t) = \exp\left[-i\,t\,H_0\right]$. To any in state $|\psi\rangle_{in} \in \mathcal{H}$, as well as to any out state it corresponds an orbit $|\psi\rangle$. However, as in the classical case, not all the state $|\psi\rangle$ in the Hilbert space $\mathcal{H}$ have an in (or out) asymptote, since there can exist bound states. Nevertheless, if the potential is sufficiently nice, all the states with an in asymptote will have also an out one, and these *scattering* states together with the bound states span the whole Hilbert space. It is convenient to introduce the Møller wave operators $\Omega_\pm$ such that:

$$\Omega_\pm = \lim_{t\to\mp\infty} U^\dagger(t)U_0(t). \tag{2.6}$$

These operators map a given in (out) state into the unique correspondent scattering state. More precisely,

$$|\psi\rangle = \Omega_+|\psi\rangle_{in} \ , \qquad |\psi\rangle = \Omega_-|\psi\rangle_{out}. \tag{2.7}$$

As we have previously emphasized, usually we are not experimentally interested in the precise trajectory of the system at all times, but we would like to understand its asymptotic properties. In particular, if the collimator prepare the system in the state $|\psi\rangle_{in}$, we would like to compute the outgoing state $|\psi\rangle_{out}$. This state is give by the action on the initial state of the S matrix

$$|\psi\rangle_{out} = S|\psi\rangle_{in} = \Omega_-^\dagger\Omega_+|\psi\rangle_{in}. \tag{2.8}$$

The S matrix is clearly an unitary operator, hence

$$S^\dagger S = \mathbf{1}. \tag{2.9}$$

The matrix elements of the S matrix gives us the transition amplitudes, so from the knowledge of the S matrix we can compute any interesting experimental quantity, e.g. the cross section. A natural basis to use in order to compute the matrix element of the S matrix is the momentum one. Notice that the Møller wave operators obey an interwinding relation:

$$H\Omega_\pm = \Omega_\pm H_0, \tag{2.10}$$

as it can be proved from

$$\begin{aligned} U(\tau)\Omega_\pm &= e^{-i\tau H} \lim_{t\to\mp\infty} e^{+itH} e^{-itH_0} = \lim_{t\to\mp\infty} e^{+i(t-\tau)H} e^{-itH_0} = \\ &= \lim_{t'\to\mp\infty} e^{+it'H} e^{it'H_0} e^{-i\tau H_0} = \Omega_\pm U_0^\dagger(\tau). \end{aligned} \tag{2.11}$$

Therefore, the states

$$|\boldsymbol{p}\pm\rangle = \Omega_\pm |\boldsymbol{p}\rangle \tag{2.12}$$

are actually eigenstates of the full Hamiltonian H with eigenvalue

$$E(\boldsymbol{p}) = \frac{\boldsymbol{p}^2}{2m}. \tag{2.13}$$

We know that, for proper normalized states, $|\psi\pm\rangle$ is the state of the system at time $t = 0$,that is evolved from (or will evolve to) the asymptotic in (or out) condition $|\psi\rangle$. Since the states (2.12) are stationary states, this is not strictly true, but holds only when we consider a normalized superposition of these states. So, if

$$|\psi\rangle = \int d\boldsymbol{p}\, \psi(\boldsymbol{p}) |\boldsymbol{p}\rangle \tag{2.14}$$

is the asymptotic condition, the state at time $t = 0$ will be:

$$|\psi\pm\rangle = \int d\boldsymbol{p}\, \psi(\boldsymbol{p}) |\boldsymbol{p}\pm\rangle \tag{2.15}$$

Nevertheless, with this caveat in mind, we will sometimes refer to $|\boldsymbol{p}+\rangle$ as the state the in the long past was a planewave $|\boldsymbol{p}\rangle$, and in a similar way of $|\boldsymbol{p}-\rangle$.

So, there are (at least) two way to look at a scattering experiment. One way (the time-dependent one) is to imagine that in a far past we have prepared our system in an incoming state $|\psi\rangle_{in}$, and the corresponding outgoing state will be determined by the action of the S operator on the in state. Otherwise, we could devote our attention to the study of the stationary state of the Hamiltonian $|\boldsymbol{p}\pm\rangle$. Since $S = \Omega_-^\dagger \Omega_+$ (2.8) the matrix elements of S are simply the inner product of these stationary states, i.e.

$$\langle \boldsymbol{q}|S|\boldsymbol{p}\rangle = \langle \boldsymbol{q} - |\boldsymbol{p}+\rangle. \tag{2.16}$$

In our arguments we will rely on both these approaches.

2.2 What is quantum integrability?

In classical mechanics (see for example [36, 37]) integrability is a well defined and established concept. Let us consider an autonomous system with n degrees of freedom: its motion is thus described by a set of $2n$ equations

$$\dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}}, \qquad \dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}}, \tag{2.17}$$

where $H(\mathbf{p}, \mathbf{q})$ is the Hamiltonian of the system. Generally speaking, in order to integrate a system of $2n$ differential equations we should know $2n$ integral of motions. However, the canonical structure of the equations of motions implies a much stronger result: if the system admits n integrals of motion, it can be explicitly integrated by quadratures. More precisely, we have the following theorem (by Liouville and Arnol'd). Let us assume that we are given n functions $F_1(\mathbf{p}, \mathbf{q}), \ldots, F_n(\mathbf{p}, \mathbf{q})$ in involution, that means their Poisson brackets[1] are vanishing

$$\{F_i, F_j\} = 0. \tag{2.18}$$

Let us consider a level set of these of these functions:

$$M_{\mathbf{f}} = \{(\mathbf{p}, \mathbf{q})\colon F_i(\mathbf{p}, \mathbf{q}) = f_i, i = 1, \ldots, n\}, \tag{2.19}$$

and let us further assume that these functions F_i are independent on $M_{\mathbf{f}}$, i.e. the linear forms dF_i are linearly independent on each point of $M_{\mathbf{f}}$. Then, we have that:

1. $M_{\mathbf{f}}$ is a smooth manifold, invariant under the phase flow with Hamiltonian F_1.

2. If the manifold is compact and connected, then it is diffeomorphic to the n-dimensional torus

$$T^n = \{(\varphi_1, \ldots, \varphi_n) \bmod 2\pi\}. \tag{2.20}$$

3. The phase flow determines a conditionally periodic motion, i.e. in angular coordinates $(\varphi_1, \ldots, \varphi_n)$:

$$\frac{d\boldsymbol{\varphi}}{dt} = \boldsymbol{\omega}, \qquad \boldsymbol{\omega} = \boldsymbol{\omega}(\mathbf{f}). \tag{2.21}$$

[1] We remind that the Poisson bracket of two functions of the phase space is $\{f, g\} = \sum_i \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}$

4. The canonical equations can be integrated by quadratures.

The last point can be achieved since, under the hypotheses of the theorem, we can pass through a (time independent) canonical transformation to the action-angle variables $(\boldsymbol{I}, \boldsymbol{\varphi})$, where the action variables $\boldsymbol{I}$ are functions of the n integrals of motion F_i and thus constant in time, while the angle variables $\boldsymbol{\varphi}$ describe the motion on the torus:

$$\boldsymbol{I(t)} = \boldsymbol{I}(0), \qquad \boldsymbol{\varphi}(t) = \varphi(0) + \boldsymbol{\omega} t. \tag{2.22}$$

Notice that if the manifold $M_{\mathbf{f}}$ is compact but not connected, the theorem holds for every connected component.
So, summarizing, we have that in the classical case the existence of n independent integrals of motion in involution provides us with a constructive way (the action-angle variables) to integrate explicitly the equations of motion. Moreover, this property characterized qualitatively the motion of the system, since it spans the invariant manifold $M_{\mathbf{f}}$ and not the whole hypersurface with constant energy. Finally, all the known examples of dynamical systems that have been integrated to the present day have a number of independent constants of motion in involution equal to the number of the degrees of freedom.

However, a proper definition of integrability in the quantum case is more subtle. A nice discussion of this point can be found in [19, 38]. Naively, one would be tempted to generalized the classical definition substituting Poisson brackets with quantum commutators. Therefore, a system would be called integrable if there exist as many independent commuting operators F_i as the number of the degrees of freedom. However, what is the number of the degrees of freedom for a quantum systems without a classical analog, as a chain of N spins? Should we assume that it is the dimension of the Hilbert space D, that scales exponentially in N, or should we suppose that the number of the degrees of freedom is proportional to N? Moreover, it is not so clear what is a proper definition of independence. One would assume that, as in the classical case, a suited definition is the functional one. However, let us assume that F_1, one of the commuting operators, is non degenerate. Since commuting operators can be simultaneously diagonalized, i.e.

$$F_i = \sum_{j=1}^{D} \lambda_i^j P_j, \tag{2.23}$$

where P_j is a set of orthogonal projectors, it is not so surprising that any F_i can be written as a polynomial of F_1:

$$F_i = \sum_{j=1}^{D} \lambda_i^j \prod_{k \neq j} \frac{F_1 - \lambda_1^k}{\lambda_1^j - \lambda_1^k}. \tag{2.24}$$

Indeed, the non degeneracy of F_1 allow us to invert (2.23)

$$P_j = \prod_{k \neq j} \frac{F_1 - \lambda_1^k}{\lambda_1^j - \lambda_1^k}. \tag{2.25}$$

So, a more sound definition of independence is the algebraical one: therefore, a maximal set of independent integrals of motion in involution will contain D elements (the number of diagonal entries of a $D \times D$ matrix). Examples of such a set would be the power of the Hamiltonian H^j, $j = 1, \ldots, D$, or the projectors P_j-but these constants of motion exist for every quantum Hamiltonian!
Moreover, the explicit knowledge of a (maximal) set of integral of motion doesn't allow us to solve the system, i.e. determine the eigenvalues and the eigenfunctions of the Hamiltonian.

Therefore, it seems that a proper definition of quantum integrability cannot be obtained by a straightforward generalization of the classical one. So, let us adopt a different strategy: instead of trying to extend the classical definition of integrability, let us focus our attention directly on solvable quantum many body systems and let us try to identify the physical property that enable us to write down such a solution. From now on, we will consider only one dimensional systems that support scattering. For simplicity, let us start by focusing our attention on a non-relativistic system with only one kind of particle with mass m -we will relax these assumptions in the next few pages. The particles interact through a short distance two body potential $V(|x_1 - x_2|)$l and we assume that the total number of particle is conserved. If we have only one particle, the eigenfunctions of our system are simply plane waves

$$\psi_k(x) = e^{ikx}, \tag{2.26}$$

with momentum k and energy

$$E(k) = \frac{k^2}{2m}. \tag{2.27}$$

Let us consider the two particle problem. At time $t = -\infty$, the particles (with an almost well defined momentum k_1 and k_2, as well as position x_1

and x_2) are far apart, hence they can be described by plane waves - in the spirit of the discussion at the end of sec. 2.1. So, the asymptotic energy and momentum of the state is simply:

$$\begin{aligned} P &= k_1 + k_2, \\ E &= E(k_1) + E(k_2). \end{aligned} \tag{2.28}$$

Let us assume now that the $x_1 < x_2$ while $k_1 > k_2$. Since the faster particle is "behind" the slower one, the two particles will collide. So, at time $t = +\infty$, the two wavepackets will be far apart, but with asymptotic momentum and energy

$$\begin{aligned} P &= k'_1 + k'_2, \\ E &= E(k'_1) + E(k'_2), \end{aligned} \tag{2.29}$$

where $k'_1 k'_2$ are the outgoing (asymptotic) momenta of the two particles. However,the only possible solution to these two equations is that the outgoing momenta are equal to the incoming ones. Notice that this is true only in 1+1 dimensions: if the number of the spatial dimensions is greater then one the momentum and the energy conservations do not imply the identity of the incoming and outgoing momenta. We would like also to emphasize that a crucial role is played by the *locality* of the momentum and energy operators, that imply that their spectrum is the sum of the single particle contributions. As we have discussed in sec. 2.1, from the plane waves $|k_1 k_2\rangle$, that are the eigenfunctions of the free Hamiltonian, we can build a stationary state for the interacting Hamiltonian H $|k_1 + k_2 +\rangle$. In the asymptotic region, $x_1 \ll x_2$, the corresponding wavefunction is

$$\psi(x_1, x_2) = \exp[i(k_1\, x_1 + k_2\, x_2)] + S(k_1, k_2) \exp[i(k_2\, x_1 + k_1\, x_2)], \;\; x_1 \ll x_2, \tag{2.30}$$

while the wavefunction in the region $x_2 \gg x_1$ is determined by the quantum statistics of the particles (bosonic or fermionic). In eq. (2.30) appears the (two particles) S matrix

$$S(k_1, k_2) = \frac{\langle k_2 k_1 | S | k_1 k_2 \rangle}{\langle k_2 k_1 | k_1 k_2 \rangle}, \tag{2.31}$$

where $\langle k_2 k_1|$ is the dual vector to $|k_1 k_2\rangle$[2] . $S(k_1, k_2)$ will be a central object in our investigation, and we will analyze its properties in much more details in the following sections. Here, we would like to emphasize only that, in

[2]Since the particles are identical, we cannot distinguish a reflection amplitude and a transmission one.

this simplified case, the S matrix $S(k_1, k_2)$ is simply a complex number of modulus one (due to the unitarity of S), and that its dependence from the asymptotic momenta is actually restricted by the symmetry of the potential. For example, Galilean invariance implies that $S = S(k_1 - k_2)$. Let us consider now the case of three particles. As usual, we imagine to prepare our system in a incoming state with wavepackets of almost well defined momentum far apart. The total momentum and energy are now

$$\begin{aligned} P &= k_1 + k_2 + k_3, \\ E &= E(k_1) + E(k_2) + E(k_3), \end{aligned} \tag{2.32}$$

where $k_1 > k_2 > k_3$ are the incoming asymptotic momenta, while the wavepackets are spatially ordered in the opposite way. However, in this case, the outgoing momenta are not uniquely determined by the the energy and momentum conservation. Therefore, the stationary wavefunction in the asymptotic region $x_1 \ll x_2 \ll x_3$ will be the sum of two pieces

$$\psi(x_1, x_2, x_3) = \psi_{BETHE}(x_1, x_2, x_3) + \psi_{DIFF}(x_1, x_2, x_3). \tag{2.33}$$

The first part, the *Bethe* wavefuction, is simply a linear combination of plane waves

$$\psi_{BETHE}(x_1, x_2, x_3) = \sum_P \psi(P) \exp[i(k_{p_1} x_1 + k_{p_2} x_2 + k_{p_3} x_3)], \tag{2.34}$$

where the sum runs on all the permutations of three objects. First of all, let us clarify that this Bethe wavefuction is always present. Indeed, we can imagine to realize a scattering experiments where the particles collides only in pairs: this physics is described by the Bethe wavefunction. The amplitudes $\psi(P)$ are related by the two particles S matrix, e.g.

$$\psi(213) = \psi(123)\, S(k_1 - k_2). \tag{2.35}$$

However, we will have also a *diffractive* wave function, that describes the genuine three body collisions

$$\psi_{DIFF}(x_1, x_2, x_3) = \int_{\substack{k_1' < k_2' < k_3' \\ P,E}} dk_1'\, dk_2'\, dk_3'\, S_3(k_1', k_2', k_3') \exp[i(k_1' x_1 + k_2' x_2 + k_3' x_3)], \tag{2.36}$$

where the 3 body scattering amplitude $S_3(k_1', k_2', k_3')$ appears. This diffractive term is physically important, since it leads to the relaxation of momenta. However, it may happen that this diffractive terms is absent: in this case the eigenfunctions of the Hamiltonian (in the asymptotic region) are simply given

by the Bethe wavefunction. We can repeat this analysis for the $4, 5, \ldots$ particles wavefunctions: energy and momentum conservations do not rule out the possibility of having a diffractive term. However, for some very special potential, it may happen that this diffractive term is always absent. In this case, every scattering event can be factorized in a sequence of two body collisions, and therefore the eigenfunctions in the asymptotic region are of the Bethe form (2.34). In this case it is possible to"solve" the model, determining e.g. its spectrum and its thermodynamical properties. Therefore, the factorizability of the scattering seems to be a very sound definition of quantum integrability, and this is the definition we will adopt in the following. Even if this definition is the best one available, we would like to emphasize that the search for more general definitions of integrability is still a quite active area of research (see for example [38] and [39]).
Finally, let us consider briefly a multi-component system: is our analysis still valid? The answer is yes, but with an important *caveat* . If we have more than one species of particles, the (two particles) S matrix will have also indices labeling the incoming and outgoing particles

$$S_{a_1 a_2}^{b_1 b_2}(k_1, k_2) = \frac{\langle b_2(k_2) b_1(k_1)|S|a_1(k_1) a_2(k_2)\rangle}{\| |a(k_1) a_2(k_2)\rangle \| \, \| |b_1(k_1) b_2(k_2)\rangle \|}, \tag{2.37}$$

where a and b distinguish between different kinds of particles. So, if we have d different species of particles, the S matrix (2.37) at assigned momenta is a d×d matrix. Analyzing the Bethe wavefunction, we have stated that the its amplitude are related by the two particles S matrix (2.35). However, there are several ways to go from the configuration 123 to 321 by transpositions: for example, $123 \to 213 \to 231 \to 321$ or $123 \to 132 \to 312 \to 321$. One could wonder if these two different ways lead to the same relation between $\psi(123)$ and $\psi(321)$, as they should. For a system with only one kind of particle this is always the case, since the S matrices are commuting numbers. Instead, for multicomponent systems, the S matrix, hence it must satisfy a commutation equation

$$S_{a_1 a_2}^{c_1 c_2}(k_1, k_2) S_{c_1 a_3}^{b_1 c_3}(k_1, k_3) S_{c_2 c_3}^{b_2 b_3}(k_2, k_3) = S_{a_2 a_3}^{c_2 c_3}(k_2, k_3) S_{a_1 c_3}^{c_1 b_3}(k_1, k_3) S_{c_1 c_2}^{b_1 b_2}(k_1, k_2). \tag{2.38}$$

These equation is know as the Yang-Baxter equations (fig. 2.2), and it's physical meaning is quite transparent: it implies that it is possible to factorize any 3 body collision in a sequence of two-particle scattering events, and the order of this factorization doesn't matter. We will discuss in more details multi-components systems in the next section.

2.3 Integrable Field Theories

In the previous section, we have discussed extensively the concept of quantum integrability. We have seen it is not a naive translation of the classical concept, but it is instead related to the factorizability of the scattering processes. For two particles, energy and momentum conservation implies that the set of the final momenta is equal to the set of the initial ones. Instead, for three body scattering, these conservations laws are not sufficient to rule out diffraction. An insightful reader could guess that, if we have additional non trivial *local* (hence additive) conserved quantities, diffraction could be forbidden. These is actually what happens for integrable field theories, as we will argue in the next few pages.

Integrable field theories are a very special class of 1+1 dimensional relativistic field theory that can be solved exactly (see [20, 40, 41] for a review). These theories are characterized by an infinite set of local conserved currents

$$\partial_\mu J^\mu_s(x) = 0, \tag{2.39}$$

where s is a spin index that assumes infinite many values. Therefore, integrating the spatial component of these currents, we obtain a set of nontrivial conserved quantities in involution

$$Q_s = \int dx_1 J^1_s(x_0, x_1), \qquad \frac{d}{dt} Q_s = 0, \qquad [Q_s, Q_{s'}] = 0. \tag{2.40}$$

Since these conserved quantities commute, they can be simultaneously diagonalized: this is realized on the particle basis. Instead of using the momentum as a quantum number, it is convenient to introduce the rapidity θ, related to the (single particle) energy and momentum in the following way:

$$E = m_a \cosh(\theta), \qquad p = m_a \sinh(\theta). \tag{2.41}$$

Notice that we will always consider massive field theory, hence the interaction is short-range. The simplest example of conserved quantities are the spin one light-cone momenta P and $\overline{P}$, whose action on the single particle states is

$$P|a(\theta)\rangle = m_a e^\theta |a(\theta)\rangle, \qquad \overline{P}|a(\theta)\rangle = m_a e^{-\theta} |a(\theta)\rangle. \tag{2.42}$$

Instead, the action of the conserved quantities Q_s is

$$Q_s|a(\theta)\rangle = q^{(s)}_a e^{s\theta} |a(\theta)\rangle, \tag{2.43}$$

hence $Q_{|s|}$ transform as s copies of P, while $Q_{-|s|}$ transform as s copies of $\overline{P}$. We are now ready to discuss the consequences of the existence of these non trivial quantities on the scattering properties of the theory. Nicely, it turns out that:

- All the scattering events are *elastic*: the number of particles is conserved, the set of the initial masses coincides with the final one, and the set of the incoming momenta is equal to the set of the outgoing ones.
- Scattering is factorizable: any n particle scattering event can be decomposed in a sequence of 2 particle collisions. Since the S matrix obey the Yang-Baxter equations (fig.2.2), the order of this decomposition does not matter.

Let's now argue why these remarkable properties are true. Since the conserved quantities Q_s comes from local conservation laws, they are additive: if we take a n particles state composed by wavepacket of almost well defined momentum and far apart from each other, the action of the operator Q_s on this state will be the sum of the action on each wavepacket. So:

$$Q_s|a_1(\theta_1)\dots a_n(\theta_n)\rangle = \left(\sum_{j=1}^{n} q_{a_j}^{(s)} e^{s\theta_j}\right)|a_1(\theta_1)\dots a_n(\theta_n)\rangle. \tag{2.44}$$

We prepare the system in such a state and we let it evolve. At time $t = +\infty$ the system will be in another asymptotic state, with wavepackets of almost well defined momentum far apart form each other. So, we will have:

$$Q_s|b_1(\theta'_1)\dots b_m(\theta'_n)\rangle = \left(\sum_{j=1}^{m} q_{b_j}^{(s)} e^{s\theta'_j}\right)|b_1(\theta_1)\dots b_m(\theta_m)\rangle. \tag{2.45}$$

But since Q_s is a conserved quantities, we have that:

$$\sum_{j=1}^{n} q_{a_j}^{(s)} e^{s\theta_j} = \sum_{j=1}^{m} q_{b_j}^{(s)} e^{s\theta'_j}. \tag{2.46}$$

These equations must hold for infinite many s: therefore, we are forced to conclude that:

$$n = m, \qquad \{\theta_i\} = \{\theta'_i\}, \qquad \{q_{a_j}^{(s)}\} = \{q_{b_j}^{(s)}\}. \tag{2.47}$$

These conservation laws are often called an infinite set of "close to free" conservation laws, in the sense that in the asymptotic states these conservation laws tend to the ones of free theory, that impose the conservation of the individual momentum of each particle.
Let us discuss now the factorization property. Here, the crucial point is to understand that the action $\exp(-i\alpha Q_s)$, with $|s| > 1$, on a single particle

wavepacket, translate the center of the wavepacket of an amount that depends on its momentum. Indeed, let's consider a Gaussian wavepacket of almost well defined momentum p_1 and position x_1

$$\psi(x) \sim \int dp\, e^{-a^2(p-p_1)^2} e^{ip(x-x_1)}. \tag{2.48}$$

If we act with an operator giving a momentum dependent phase factor $e^{-i\phi(p)}$, the wavefunction becomes

$$\psi(x) \sim \int dp\, e^{-a^2(p-p_1)^2} e^{ip(x-x_1)-i\phi(p)}. \tag{2.49}$$

The main contribution to this integral come from the region where $p \approx p_1$.Therefore, we conclude that the "momentum" of the wavepacket doesn't change, while the center is shifted to $\tilde{x}_1$

$$\tilde{x}_1 = x_1 + \phi'(p_1). \tag{2.50}$$

So, if we apply the operator $\exp(-i\alpha Q_s)$, $|s| > 1$ to an asymptotic wavepacket with 3 particles with different momenta, we will translate each particle by a different amount. In this way,we can actually alter the temporal order in which these particles collide. But Q_s is a conserved quantity, hence

$$\langle out|S|in\rangle = \langle out|e^{i\alpha Q_s} S e^{-i\alpha Q_s}|in\rangle. \tag{2.51}$$

So, this argument lead us to the the Yang Baxter equation

$$S_{a_1a_2}^{c_1c_2}(\theta) S_{c_1a_3}^{b_1c_3}(\theta+\theta') S_{c_2c_3}^{b_2b_3}(\theta') = S_{a_2a_3}^{c_2c_3}(\theta') S_{a_1c_3}^{c_1b_3}(\theta+\theta') S_{c_1c_2}^{b_1b_2}(\theta), \tag{2.52}$$

whose pictorial representation is shown in fig. 2.2. Here the argument of the S matrix is the difference of the incoming rapidities. Relativistic invariance implies that the S matrix can depend only on Lorentz invariant quantities, such as the difference of the two incoming rapidities, since after a Lorentz boost each rapidity is shifted by a constant. We will come back to this point in the next section. Of course, our argument supporting the Yang-Baxter equation is quite heuristic, but it can be refined in a rigorous proof [42]. It turns out that we do not actually need an infinite set of additive conserved quantities Q_s: two additive quantities Q_s with spin greater than one are sufficient to derive the Yang Baxter equations. Interestingly, the existence of these two non trivial integrals of motion is also sufficient to prove the elasticity of the scattering. The argument is quite different from the one we gave before and it is based on the "momentum dependent translation" operators $\exp(-i\alpha Q_s)$. Essentialy, if the scattering is not elastic, we can act

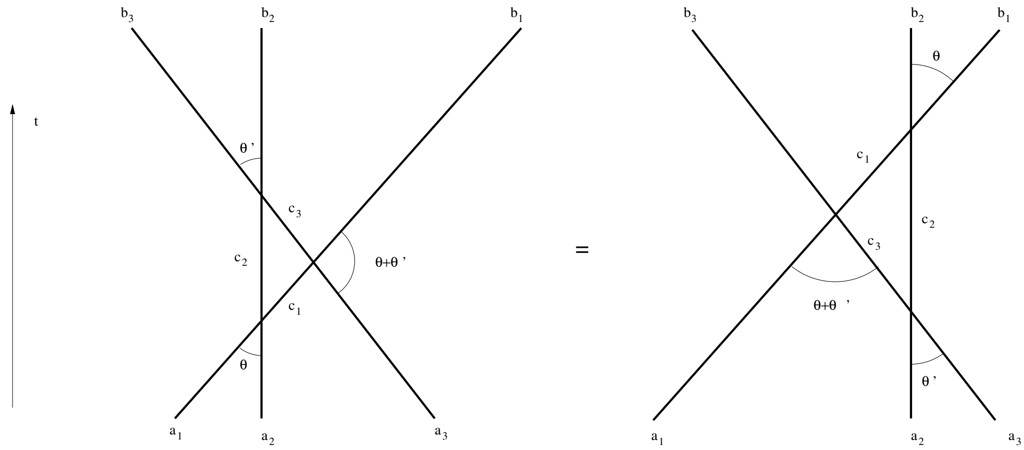

Figure 2.2: A graphical representation of the Yang-Baxter equations.

with these operators on the initial and final state and violate macrocausality. This result could be quite surprising: it turns out that only two non trivial conserved quantity are enough to guarantee the integrability of the theory. Nevertheless, it has a nice counterpart in higher dimension quantum field theories. Indeed, it was proved by Coleman and Mandula [43] in the 1967 that in three spatial dimensions the existence of just one non trivial conserved quantity transforming as a tensor of second or higher order forces the S matrix to be the identity. This fact can be grasped in the following way: in $1+1$ dimension, even if I reshuffle the positions of the particles acting with the operators $\exp(-i\alpha Q_s)$, their trajectories will always cross somewhere. Instead, if we have more dimensions to play with, we could always translate the particles in such a way they never become so close to feel their mutual interaction.

2.4 Analytical Structure of Integrable Field Theories

2.4.1 Analytical Properties of the S matrix

In the previous section, we have shown that in 1+1 dimensions it is possible to have (non trivial) relativistic field theories that are *integrable*, in the sense that scattering is factorizable. In this section, we would like to shows how we can exploit integrability to solve the theory, computing exactly the S matrix as well as the matrix elements of local operators. Quite nicely, this solution is based on the analytical properties of the S matrix in the complex plane. This set of ideas dates back to the S matrix school in particle physics. Pioneered by Heisenberg in 1941, this approach was very popular in the '50s and '60s, due to the difficulties of the study of strong interactions in a quantum field theories framework. The advocates of the S matrix theory rejected quantum field theories, proposing to predict scattering datas from the analytical properties of the S matrix. Actually, this approach was not very successful, due to the technical difficulties of the problem, and it was abandoned in the '70, when QCD provided a faithful description of strong interactions. However, this *vaste programe* turned out to be solvable in integrable field theories in 1+1 dimensions, no more in opposition to quantum field theory but as a complimentary approach.

In a relativistic field theory, the amplitude of a $2 \to 2$ particle scattering is usually parametrized in terms of the Mandelstam variables:

$$s = (p_1 + p_2)^2, \quad t = (p_1 - p_1')^2, \quad u = (p_1 - p_2')^2, \tag{2.53}$$

where p_1 and p_2 are the incoming momenta, while p_1' and p_2' are the outgoing ones[3]. However, in $1+1$ dimension only one of this variable is independent and it is customary to choose s, the energy of the center of mass. In a collision between two particles of mass m_1 and m_2, s is of course real and greater than $(m_1 + m_2)^2$. However, it is useful to continue analytically $S_{a_1 a_2}^{b_1 b_2}(s)$ to the complex plane: its analytical properties are shown in fig. 2.3.

First of all, S in a singlevalued, meromorphic function on the complex plane

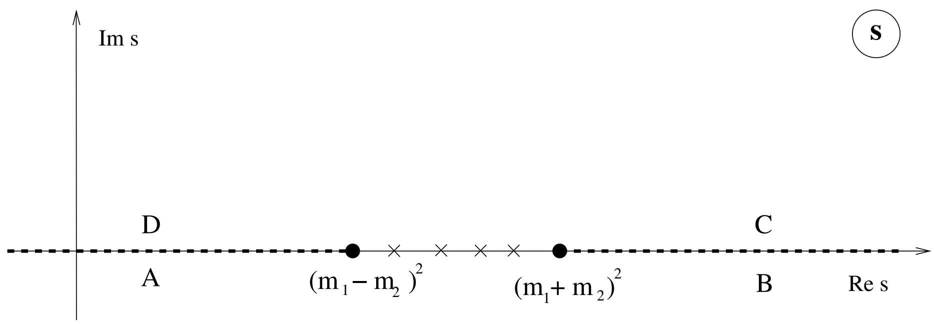

Figure 2.3: Analytical properties of the S matrix in the physical sheet.

with cuts on the real axis for $s < (m_1 - m_2)^2$ and $s > (m_1 + m_2)^2$ -the reason of these cuts will be clear in a few moments. The physical values of $S(s)$ are found just above the right hand cut (we use the obvious notation s^+) , while

[3]Here the square symbol denotes the relativistic norm.

the first sheet of the full Riemann surface for S is called the physical sheet. Moreover, S is real analytic, i.e.

$$S_{a_1 a_2}^{b_1 b_2}(s^*) = \left[S_{a_1 a_2}^{b_1 b_2}(s)\right]^* . \tag{2.54}$$

We know that S is an unitary operator, hence

$$S^\dagger S = \mathbf{1}. \tag{2.55}$$

Let us evaluate this operatorial identity on two particles asymptotic states, i.e.

$$\langle b_2(p_2) b_1(p_1)|S^\dagger S|a_1(p_1) a_2(p_2)\rangle = \delta_{a_1}^{b_1} \delta_{a_2}^{b_2}. \tag{2.56}$$

In order to evaluate this expression, we should insert a multiparticle resolution of the identity between $S^\dagger$ and S. If the energy available for the scattering s is not too large, energy conservation will forbid the creation of (real) particles in the intermediate state. However, for a generic theory, for s greater than a threshold determined by the masses of the particles of the theory, particle production will be allowed and unitarity becomes a quite complicated condition involving a sum of a large number number of scattering amplitudes. Nevertheless, integrability rules out this possibility, hence unitarity reads as

$$\left[S_{c_1 c_2}^{b_1 b_2}(s^+)\right]^* \, S_{a_1 a_2}^{c_1 c_2}(s^+) = \delta_{a_1}^{b_1} \delta_{a_2}^{b_2}, \tag{2.57}$$

or,in virtue of (2.54),

$$S_{a_1 a_2}^{c_1 c_2}(s^-) \, S_{a_1 a_2}^{c_1 c_2}(s^+) = \delta_{a_1}^{b_1} \delta_{a_2}^{b_2}. \tag{2.58}$$

This equation shows the presence of the cut for $s > (m_1 + m_2)^2$: for non integrable field theories we would have also other cuts on the real axis corresponding to the the other production thresholds.
Let us consider now the left-hand cut, that starts at $(m_1 - m_2)^2$. What is its physical meaning? This can be understood by appealing to the crossing symmetry, a property of any relativistic invariant field theory. Let us consider the two particle S matrix represented in fig. 2.4. If time flows vertically, this amplitude represents a collision between two incoming particles a_1, a_2, with momenta p_1, p_2, while the outgoing particles are b_1, b_2 with momenta p_1, p_2. However, we can imagine that time flows horizontally in this picture, and the arrows that flow backward in time represents the charge-conjugated antiparticle. The energy available for the the scattering is now $t = (p_1 - p_2)^2 = 2\,[m_1^2 + m_2^2] - s$. The crossing symmetry implies that the scattering amplitude in the crossed channel (time flows horizontally) can

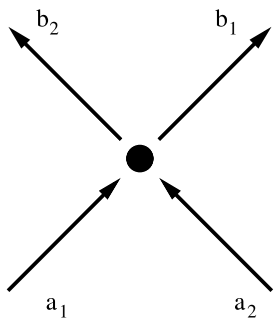

Figure 2.4: A pictorial representation of the two particle S matrix

be obtained by analytical continuation from the amplitude of the forward channel (time flows vertically), hence

$$S_{a_1 a_2}^{b_1 b_2}(s^+) = S_{a_1 \bar{b}_2}^{b_1 \bar{a}_2}(2\left[m_1^2 + m_2^2\right] - s^+), \tag{2.59}$$

where the overline denotes the charge conjugated antiparticles.

Therefore, there should be also a left-handed cut that starts at $(m_1 - m_2)^2$. It is possible to show these are square root branch points. Finally, if the theory has bound states with mass $m_k \in (|m_1 - m_2|, m1 + m2)$, our experience in perturbative field theory, as well as the quantum-mechanical theory of scattering, suggests the $S(s)$ should have poles for $s = m_k^2$. A part for the cuts (due to unitarity) and the poles for $s \in (|m_1 - m_2|, m1 + m2)$, we require that the S matrix has no other singularities: therefore, we are requiring the most analytical behavior consistent with the physical requirements discussed before.

In $1+1$ dimensions, it is consistent to study the analyticity of the S matrix not as a function of s, but instead as a function of the difference of the two incoming rapidities $\theta = \theta_1 - \theta_2$, whose relation with S is

$$s = (p_1 + p_2)^2 = m_{a_1}^2 + m_{a_2}^2 + 2 m_{a_1} m_{a_1} \cosh(\theta). \tag{2.60}$$

In this way the cuts are opened up, while the physical strip is mapped in the region $0 \leq \mathrm{Im}\theta \leq \pi$. The analytical structure of the S matrix in the θ plane is shown in fig. 2.5.

So, let us discuss how the previous relations translate in the θ plane:

- **Real Analiticity** The amplitudes $S_{a_1 a_2}^{b_1 b_2}(\theta)$are meromorphic functions of θ, real at $Re(\theta) = 0$. The only poles of the S matrix in the physical strip lies on the imaginary axis.

- **Crossing Symmetry** The physical scattering amplitude of the direct channel $a_1 a_2 \to b_1 b_2$ is given by the values of the functions $S_{a_1 a_2}^{b_1 b_2}(\theta)$ at

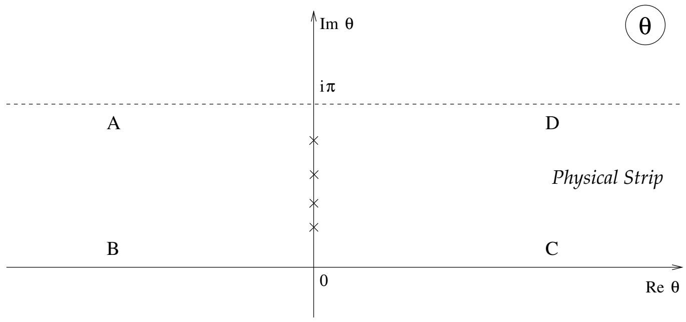

Figure 2.5: Analytical properties of the S matrix in the θ plane

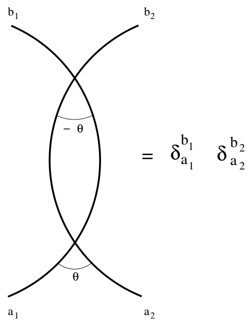

Figure 2.6: A graphical representation of the unitarity condition.

$Im(\theta) = 0$, $Re(\theta) > 0$. This amplitude is related to the cross-channel one $a_1\overline{b}_2 \to b_1\overline{a}_2$ by the relation

$$S_{a_1 a_2}^{b_1 b_2}(\theta) = S_{b_1 \overline{a}_2}^{a_1 \overline{b}_2}(i\pi - \theta). \tag{2.61}$$

- **Unitarity** The unitarity of the S matrix is expressed by the condition

$$S_{a_1 a_2}^{c_1 c_2}(\theta) S_{c_1 c_2}^{b_1 b_2}(-\theta) = \delta_{a_1}^{b_1}\, \delta_{a_2}^{b_2}, \tag{2.62}$$

whose pictorial representation is shown in fig. 2.4.1

Moreover, we remind from the previous section that the S matrix must satisfy the

- **Yang Baxter Equations**

$$S_{a_1 a_2}^{c_1 c_2}(\theta) S_{c_1 a_3}^{b_1 c_3}(\theta + \theta') S_{c_2 c_3}^{b_2 b_3}(\theta') = S_{a_2 a_3}^{c_2 c_3}(\theta') S_{a_1 c_3}^{c_1 b_3}(\theta + \theta') S_{c_1 c_2}^{b_1 b_2}(\theta), \tag{2.63}$$

whose pictorial representation is shown in 2.2.

2.4.2 The Zamolodchikov-Fadeev Algebra

We are now finally ready to introduce a convenient quasi particle basis that diagonalizes the Hamiltonian. In sec.2.1 we have discussed how it is possible to construct a basis of eigenstates of the interacting Hamiltonian from the plane waves, by the action of the Møller operators $\Omega_{\pm}$. (eq. 2.12). Moreover, we know that the inner product of these stationary states is related to the appropriate scattering amplitude (2.16). We denote with $|a_1(\theta_1)\dots a_1(\theta_n)\rangle_0$ the eigenstate of the free Hamiltonian where the particle a_i has rapidity θ_i, and let us assume that $\theta_i > \theta_j$ for $i > j$. The stationary state $\Omega_+|a_1(\theta_1)\dots a_1(\theta_n)\rangle_0$ is called an *in state*, while the stationary state $\Omega_-|a_1(\theta_1)\dots a_1(\theta_n)\rangle_0$ is called an *out state*. In $1+1$ dimension we can distinguish an in state from the out state with the same particles and rapidity simply by the ordering of the rapidities: in an in state the rapidities are ordered (from left to right) in a decrescent way, while for the out state the order is the opposite, i.e.

$$\begin{aligned}
&\textbf{IN STATE } |a_1(\theta_1)\dots a_n(\theta_n)\rangle := \Omega_+|a_1(\theta_1)\dots a_1(\theta_n)\rangle_0,\\
&\textbf{OUT STATE } |a_n(\theta_n)\dots a_1(\theta_1)\rangle := \Omega_-|a_1(\theta_1)\dots a_1(\theta_n)\rangle_0, \quad (2.64)
\end{aligned}$$

where $\theta_1 > \theta_2 > \dots > \theta_n$. This definition remind us that, before that any collisions happen, the fastest particle is to the left while the slowest one is to the right, while for $t \to +\infty$ the order will be reversed. Clearly, this notation is convenient only in 1 spatial dimension. Now, we wonder if it is possible to have a set of creation/annihilation operators $Z^a(\theta), Z^\dagger_a(\theta)$ such that, both for in and out states,

$$|a_1(\theta_1)\dots a_n(\theta_n)\rangle = Z_1^a(\theta_1)\dots Z_n^a(\theta_n)|0\rangle, \tag{2.65}$$

where the vacuum $|0\rangle$ is the state annihilated by all the $Z^a(\theta)$ operators. The answer is that is indeed possible, provided that the operators $Z^a(\theta), Z^\dagger_a(\theta)$ satisfy the Zamolodchikov-Fadeev algebra:

$$Z^{b_1}(\theta_1)Z^{b_2}(\theta_2) = S^{b_1b_2}_{a_1a_2}(\theta_1-\theta_2)Z^{a_2}(\theta_2)Z^{a_1}(\theta_1), \tag{2.66}$$

$$Z^\dagger_{a_1}(\theta_1)Z^\dagger_{a_2}(\theta_2) = S^{b_1b_2}_{a_1a_2}(\theta_1-\theta_2)Z^\dagger_{b_2}(\theta_2)Z^\dagger_{b_1}(\theta_1), \tag{2.67}$$

$$+2\pi\delta(\theta_1-\theta_2)\delta^{b_1}_{a_1}\delta^{b_2}_{a_2}. \tag{2.68}$$

So, these operators do not simply commute or anticommute. Instead, every time we exchange two of them we pick up an S matrix. Heuristically, we could say that this is a reminder of the fact that in one spatial dimension we cannot exchange two particles without bringing them close, hence the two particle interact. So, is this algebra self consistent? We have to check two things. First of all, if we exchange two operators two times in a row, do we return to the initial situation?This fact is guaranteed by the unitarity condition (2.62). The other thing that we should check is the following: any permutation can be decomposed as a sequence of transpositions, but thise decomposition is not unique.Therefore, we could wonder: if we do a permutation of a sequence of Z operators in two different way, do we get the same result? The answer to this question is yes, thanks to the Yang Baxter equation. (2.63). Finally, it is clear that the in state are transformed into the out ones by the S matrix[4] . Of course, we can also have states $|a(\theta_1)\dots a(\theta_n)\rangle$ where the rapidities are not ordered in a decrescent or crescent way. These stationary states are not in or out states in the sense defined before, but instead correspond to the situation where some particles have already scattered and some not yet.

2.4.3 Poles Structure in Diagonal Theories and The Bootstrap Principle

Unitarity (2.62) and crossing (2.61) are not sufficient to determine univoquely the S matrix. They should be supplemented also by an analysis of the poles and by the *bootstrap principle*, or nuclear democracy: it states that bound states are on the same footing of all the other particles of the theory. In order to briefly discuss the main conceptual points of this theoretical framework, we will consider only diagonal scattering theories, where all the masses are different hence any scattering process is simply a reshuffling of the momenta. In this case, the S matrix is diagonal hence the Yang-Baxter equations are trivially satisfied. Since the set of the outgoing particles is equal to the ingoing one, the S matrix depends only on two indices $S_{a_1a_2}(\theta)$. Unitarity and crossing now read as

$$S_{a_1a_2}(\theta)S_{a_1a_2}(-\theta) = \mathbf{1}, \tag{2.69}$$

$$S_{a_1a_2}(\theta) = S_{a_1\overline{a}_2}(i\pi - \theta). \tag{2.70}$$

These two conditions prove that $S_{a_1a_2}(\theta+2i\pi) = S_{a_1a_2}(\theta)$, hence the Riemann surface of the S matrix is simply a double cover of the complex plane. Now, let us assume that the $S_{a_1a_2}(\theta)$ has a pole for $\theta = iU^b_{a_1a_2}$, $U^b_{a_1a_2} \in [0, i\pi]$. This pole can be interpreted as the presence of a bound state $\overline{b}$,in the direct or in the crossed channel, whose propagator becomes on shell for $\theta = iU^b_{a_1a_2}$ (fig. 2.7) As a consequence of the presence of this pole, we have:

- The vertex function $C_{a_1a_2b}$ is non zero when a_1, a_2 and b are on shell.
- **Nuclear Democracy** For $\theta = i\,U^b_{a_1a_2}$, the intermediate particle $\overline{b}$ is on shell and survive for macroscopic times. The bootstrap principle (or

[4]This definition of the S matrix is actually slightly different from the one used in sec.2.1

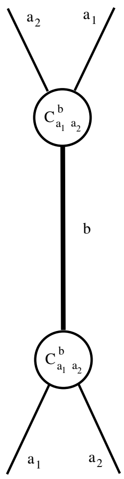

Figure 2.7: The two particle S matrix $S_{a_1a_2}(\theta)$, for $\theta \approx iU^b_{a_1a_2}$, is dominated by the contribution of the on shell propagator.

nuclear democracy) implies that this bound state is expected to be one of the other particles of the model.

- Since $s = m_b^2$ when $\theta = iU^b_{a_1a_2}$, we have the following relation between the masses

$$m_b^2 = m_{a_1}^2 + m_{a_1}^2 + 2m_{a_1}m_{a_2}\cos(U^b_{a_1a_2}). \tag{2.71}$$

A geometrical interpretation of this formula is shown in fig.2.8: the "fusion angle" $U^b_{a_1a_2}$ is the external angle of a mass triangle whose sides are the masses m_{a_1}, m_{a_2} and m_b . Since $C^b_{a_1a_2}$, poles are also present in $S_{a_1}b$ and S_ba_2 . The mass triangle shows immediately that the sum of the fusion angle, as well as the sum of the external angles of a triangle, is equal to 2π, i.e.

$$U^b_{a_1a_2} + U^{a_1}_{a_2b} + U^{a_1}_{b\,a_2} = 2\pi. \tag{2.72}$$

We are now in the position to discuss more deep consequences of the presence of the pole. First of all, let us consider the scattering processes shown in fig.2.9, where two particle a_1 and a_2 fuse in a bound state $\bar{b}$ and scatter with an external particle c. As it is shown in the picture, there are two equivalent way to have such a process.The particles a_1 and a_2 could form a bound state and then interact with the particle c, or the scattering with the external particle could happen before the formation of the bound state. Following the line of reasoning we use for the derivation of the Yang-Baxter equation,

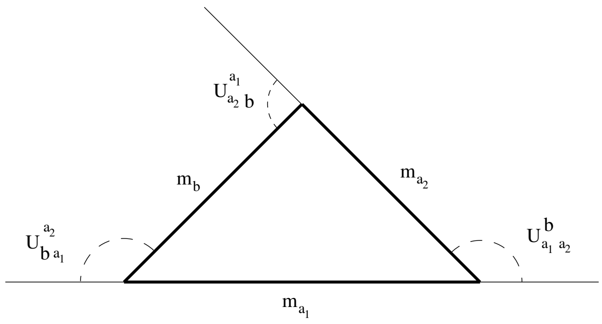

Figure 2.8: The mass triangle.

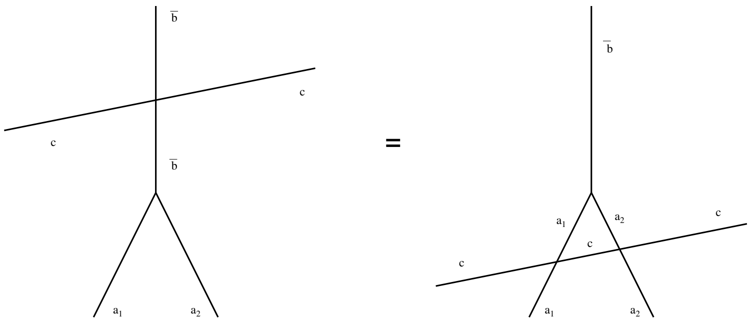

Figure 2.9: Scattering between two particles a_1, a_2, that fuse in a bound state $\bar{b}$, and an external particle c. The scattering amplitudes for the two processes shown are equal, hence the S matrix bootstrap equation holds.

we could argue that this two amplitudes are equal. Therefore, we obtain the*S matrix bootstrap equation*

$$C_{a_1 a_1 b} \neq 0 \Rightarrow S_{c\bar{b}}(\theta) = S_{ca_1}(\theta - i\overline{U}^{a_2}_{a_1 b}) S_{ca_1}(\theta + i\overline{U}^{a_1}_{a_2 b}), \tag{2.73}$$

where $\overline{U} = \pi - U$. Similarly, the fact that the particles a_1 and a_2 can form a bound state $\bar{b}$ has consequences on the conserved charges: the eigenvalues of the conserved charges relative to the bound state must be equal to the eigenvalues of the two particle states, so

$$C_{a_1 a_1 b} \neq 0 \Rightarrow q^{(s)}_{\bar{b}} = q^{(s)}_{\bar{a}_1} e^{is\overline{U}^{a_2}_{b\,a_1}} + q^{(s)}_{\bar{a}_2} e^{-is\overline{U}^{a_1}_{a_2 b}} \tag{2.74}$$

This sets of equation constitutes the conserved charge bootstrap. Given a set of masses and three-point couplings, the fusing angles are determined by the mass triangle. Therefore, the conserved charge bootstrap equations are an overdetermined set of conditions for $q^{(s)}_a$. If the only solution for a given spin s is the trivial one $q^{(s)}_a = 0\,\forall a$, then it mean that the charge with spin

s is absent in the theory under consideration.

So, the unitarity, crossing symmetry and the bootstrap principle greatly constrain the structure of the S matrix. Indeed, it is sometimes possible, by trial and error, to obtain the exact S matrix. One starts with an initial guess for the S matrix, search for poles, infer three-particles vertex functions, apply the bootstrap principle to deduce further S matrix elements and so on. If the process closes on a finite set of particles, then we have obtained the S matrix a part for the so-called CDD ambiguity, i.e. the multiplication of the S matrix for a function $\Phi(\theta)$ that satisfies

$$\Phi(\theta) = \Phi(i\pi - \theta), \qquad \Phi(\theta)\Phi(-\theta) = 1. \tag{2.75}$$

The physically meaningful S matrix is identified with the "minimal solution", i.e. the one that has the less number of singularity in the complex plane. Moreover, since the charge bootstrap equations provide a set of of spins s for which we have non trivial conserved charges, we can often use this information to identify the corresponding field theory, described by a Lagrangian or by a perturbed CFT.
As a final remark, we would like to emphasize that the pole structure of a generic (i.e. non diagonal) integrable field theory could be much more complex, since in $1+1$ dimension not all the poles have an interpretation as bound states. However, on the one hand this phenomenon is quite understood [44]. On the other hand, in this thesis we will never consider non diagonal integrable field theories, so this extra complication will play no role in the subsequent chapters.

2.4.4 Form Factors

Nicely, the analytical structure of integrable field theories not only allow us to compute the S matrix, but also the matrix elements of *local* or *semi-local* operators. A thoughtful analysis of these matrix elements could be found in [45]. Here, we would like only to summarize briefly their main properties. For simplicity, let us consider a theory with only one particle. The basic objects of the theory are the *form factors*

$$F_n^{\mathcal{O}}(\theta_1, \ldots, \theta_n) = \langle 0|\mathcal{O}(0,0)|\theta_1, \ldots, \theta_n\rangle. \tag{2.76}$$

First of all, we would like to emphasize that these form factors are equivalent to the more general matrix elements, thanks to the crossing symmetry. Indeed,

$$\langle\theta'_m \ldots \theta'_1|\mathcal{O}(0,0)|\theta_1 \ldots \theta_n\rangle = F_{n+m}^{\mathcal{O}}(\theta_1, \ldots, \theta_n, \theta'_1 - i\pi, \theta'_m - i\pi), \tag{2.77}$$

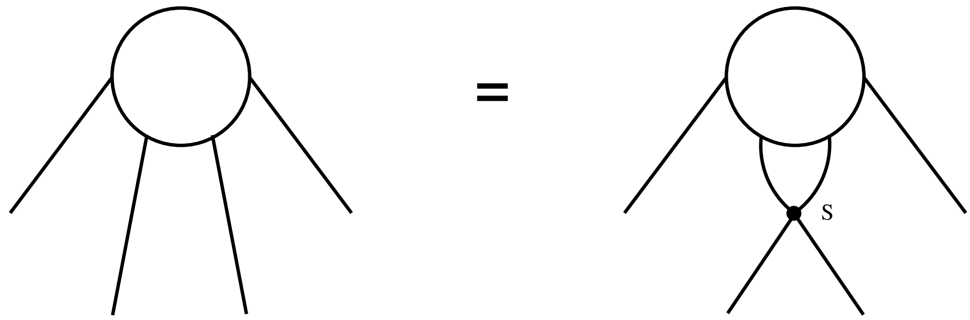

Figure 2.10: A graphical representation of the symmetry property (2.79)

if $\theta'_i \neq \theta'_j \, \forall i,j$. Otherwise, if $\theta'_i = \theta_j$ for some i,j, it is still possible to write down a general matrix element in terms of the form factors, but we should also take into account the *disconnected contributions*. These disconnected contribution could lead to divergencies whose regularization is non trivial. Since these divergencies will play a major role in our discussion of quantum quenches in integrable field theories, we prefer to discuss this formula in details in the next chapter. Obviously, since the quasiparticle basis diagonalizes both the energy and the momentum, from $\langle \mathcal{B}|\mathcal{O}(0,0)|\mathcal{A}\rangle$, where $\mathcal{A}$ and $\mathcal{B}$ are two sets of rapidities, we can obtain $\langle \mathcal{B}|\mathcal{O}(t,x)|\mathcal{A}\rangle$ simply by a translation in time and space.
Let us state now the set of equations that the form factors satisfy.For an operator of spin s, the relativistic invariance implies that

$$F_n^{\mathcal{O}}(\theta_1+\Lambda,\dots,\theta_n+\Lambda) = e^{i\Lambda s} F_n^{\mathcal{O}}(\theta_1,\dots,\theta_n). \tag{2.78}$$

For simplicity, let us consider now only a scalar operator, whose form factors depend only on the differences of rapidities. We have

- **Symmetry Property**

  $$F_n^{\mathcal{O}}(\theta_1,\dots,\theta_i,\theta_{i+1},\dots,\theta_n) = F_n^{\mathcal{O}}(\theta_1,\dots,\theta_{i+1},\theta_i,\dots,\theta_n) S(\theta_i-\theta_{i+1}). \tag{2.79}$$

  A graphical representation of this formula is shown in fig.2.10 .

- **Analytical continuation**

  $$F_n^{\mathcal{O}}(\theta_1+2\pi i,\dots,\theta_n) = e^{2\pi i\gamma} F_n^{\mathcal{O}}(\theta_2,\theta_n,\theta_1) \tag{2.80}$$

  where γ is the semi-local index of the operator O with respect to the operator that creates the particles. A graphical interpretation of this formula is shown in (2.11)

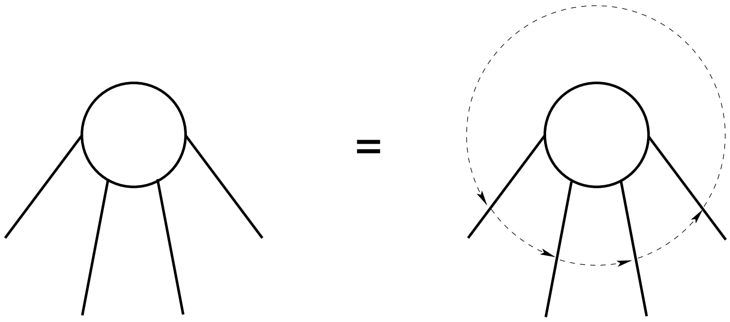

Figure 2.11: A graphical representation of (2.80)

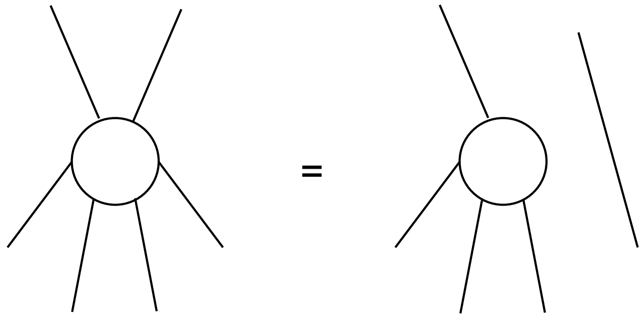

Figure 2.12: A pictorial representation of the pole structure of the form factors.

- **Pole structure**

$$-i \underset{\tilde{\theta}\to\theta}{Res} F^{\mathcal{O}}_{n+2}(\tilde{\theta},\theta,\theta_1,\ldots,\theta_n) = \left(1 - e^{2\pi i\gamma}\prod_{i=1}^{n} S(\theta-\theta_i)\right) F^{\mathcal{O}}_n(\theta_1,\ldots,\theta_n), \tag{2.81}$$

whose graphical representation is shown in (2.12).

While the symmetry property (2.79) is quite evident, formulas (2.80) and (2.81) are somehow a little more elusive. However, a strong argument in their support could be build [45], similar to the LSZ reduction formulas. Moreover, it is possible to prove that, if we assume the validity of these axioms, then it follows that local (or semi-local) operators commute at space-like distance. Since commutation at space-like distances is a fundamental property of local operators in quantum relativistic field theories, this is indeed a strong evidence in support of these assumptions. The final test for these axioms is the great body of results build from them in the last 20 years, whose correctness has been checked beyond any doubt.
As for the S matrix, it is possible to solve explicitly this set of equations,

thus obtaining an expression for the matrix elements of a given theory. As a final remark, we would like to emphasize that in integrable field theories with bound states the pole structure of the form factors is more rich. However, in the next chapters we will not consider theories with bound states.

2.4.5 Some Simple Examples

The discussion of integrable field theories in the previous sections has been quite abstract and general. We have focused our attention on the universal properties of the integrable field theories, and we have never discussed a specific model. Our reader could wonder if there exists at least *one* integrable theory. This doubt is legitimate: actually, we have seen that in higher dimensions there is a no-go theorem that forbids the existence of interacting integrable field theories. Here, we will briefly present two simple integrable field theories.

Ising Model It is well known that in the Hamiltonian limit the two-dimensional Ising model at zero external magnetic field can be mapped in a quantum spin chain in 1 dimension,

$$H = \sum_i \sigma_i^z \sigma_{i+1}^z + h\sigma_i^x, \tag{2.82}$$

know as the quantum Ising model in a transverse field. Thanks to the Jordan-Wigner and Bogoliubov transformation, this Hamiltonian is equivalent to a free fermionic theory. In the scaling limit, these fermionic theory become relativistic invariant, hence its Hamiltonian density is

$$\mathcal{H}(x) = \frac{i}{4\pi}\left[\psi(x)\partial_x\psi(x) + \overline{\psi}(x)\partial_x\overline{\psi}(x) - im\psi(x)\overline{\psi}(x)\right]. \tag{2.83}$$

There is only one particle with mass $m \sim T - T_c$, and the S matrix is simply $-\mathbf{1}$. We know that there are two phases in this model,$T > T_c$ and $T < T_c$, related by the Kramers-Wanier duality. Let us choose the disordered phase, hence $T > T_c$. At first sight, this theory could seem trivial, since it is free. However, this is not the case, since the order parameter σ is a highly non trivial operator, semi-local respect to the quasiparticle basis. The physical interesting operators are of course the order parameter σ, its dual μ and the "energy-density" $\epsilon \sim \sigma_x$. Their form factors are:

$$F_\epsilon(\theta_1, \ldots, \theta_n) = \begin{cases} -\, i\, 2\, \pi\, m\, \sinh(\frac{\theta_1 - \theta_2}{2}) & \text{if } n = 2 \\ 0 & \text{otherwise} \end{cases}, \tag{2.84}$$

$$F_\sigma(\theta_1, \ldots, \theta_n) = \begin{cases} \overline{\sigma}\, i^{(n-1)/2} \prod_{l<m}^{n} \tanh(\frac{\theta_l - \theta_m}{2}) & \text{if } n \text{ is odd} \\ 0 & \text{otherwise} \end{cases}, \tag{2.85}$$

$$F_\mu(\theta_1, \ldots, \theta_n) = \begin{cases} \overline{\sigma}\, i^{n/2} \prod_{l<m}^{n} \tanh(\frac{\theta_l - \theta_m}{2}) & \text{if } n \text{ is even} \\ 0 & \text{otherwise} \end{cases}, \tag{2.86}$$

where $\overline{\sigma} = 2^{\frac{1}{3}}\, e^{-\frac{3}{4}}\, A^3\, m^{\frac{1}{4}}$ and $A = 1.282427\ldots$ is the Glasher constant[1].

Sinh Gordon Model. The Sinh-Gordon model is described by the Lagragian

$$\mathcal{L} = \frac{1}{2}\partial_\mu \varphi(x) \partial^\mu \varphi(x) - \frac{m_0^2}{g^2} \left(\cos(g\varphi(x) - 1\right), \tag{2.87}$$

where φ is a bosonic real scalar field, m_0 a mass parameter and g an adimensional coupling constant. This model has only one particle of mass M, with

$$M^2 = m_0^2 \frac{\sin(\pi\alpha)}{\pi\alpha}, \tag{2.88}$$

where

$$\alpha = \frac{g^2}{8\pi + g^2}. \tag{2.89}$$

The S matrix of the model is

$$S(\theta) = \frac{\sinh(\theta) - i\sin(\pi\alpha)}{\sinh(\theta) + i\sin(\pi\alpha)}, \tag{2.90}$$

that correctly has no pole in the physical strip.Notice that $S(0) = -\mathbf{1}$ for $g \neq 0$. This is a simple example of a more general fact: all the integrable field theories except the free bosonic ones have $S(0) = -\mathbf{1}$, and so the elementary excitations have a fermionic nature, regardless of the statistics of the fields that appears in the Lagrangian.

The physically interesting operators are φ and its powers. All their information can be incorporated the vertex operator $e^{kg\phi}$, where k is a number.The n particles form factor can be expressed as:

$$F_n(\theta_1, \ldots, \theta_n) = [k] \left(\frac{4\sin(\pi\alpha)}{\mathcal{N}}\right)^{\frac{n}{2}} \det M_n(k) \prod_{i<j}^{n} \frac{F_{min}(\theta_i - \theta_j)}{x_i + x_j}, \tag{2.91}$$

where:

[1] Here we are using the conformal normalization of the operators, such as in the limit $x \to 0$ $\langle \phi(x, 0)\phi(0, 0)\rangle \sim \frac{1}{|x|^{4\Delta}}$ where Δ is the conformal dimension of the field.

- $F_{min}(\theta) = \mathcal{N} \exp \left[4 \int_0^{+\infty} \frac{dt}{t} \frac{\sinh(\frac{t\alpha}{2}) \sinh(\frac{t(1-\alpha)}{2})}{\sinh(t) \cosh(\frac{t}{2})} \sin^2 \left(\frac{t\hat{\theta}}{2\pi} \right) \right]$,
  where $\hat{\theta} = i\pi - \theta$ and $\mathcal{N} = F_{min}(i\pi)$. F_{min} is the part of the form factor that is reminiscent of the S matrix structure of the theory, since

  $$F_{min}(\theta) = S(\theta) F_{min}(-\theta),$$

  while all the other factors in (2.91) are symmetric under a permutation of the θ_i.

- $x_i = e_i^{\theta}$.

- $[k] = \frac{\sin(k\pi\alpha)}{\sin(\pi\alpha)}$.

- $M_n(k)$ is a $(n-1) \times (n-1)$ matrix, whose elements are symmetric polynomials in the n variables x_i defined above. More precisely, let us introduce the symmetric polynomial $\sigma_k^{(n)}$ of n variables $x_1, \ldots, x_n$ and total degree k defined as

  $$\sigma_k^{(n)}(x_1, \ldots, x_n) = \sum_{i_1 < \ldots < i_k}^{n} x_{i_1} \ldots x_{i_k}.$$

  Therefore, we have that

  $$[M_n(k)]_{ij} = \sigma_{2i-j}^{(n)}(x_1, \ldots, x_n) \times [i - j + k]$$

2.5 Boundary Integrable Field Theories

Integrable field theories are a very special class of 1+1 dimensional relativistic field theories, characterized by an infinite set of local conserved currents

$$\partial_\mu J_s^\mu(x) = 0. \tag{2.92}$$

As we have thoroughly discussed in the previous sections, the existence of these infinite many integrals of motions greatly constrains the dynamics, since only *elastic* scattering is allowed. In this section we would like to analyze what happens in presence of a boundary. For simplicity, we consider a semi-infinite system with only one boundary. So, our system is defined in the semiplane $\mathcal{S} = \{(x, y) \in \mathbf{R}^2, x \leq 0\}$. Two different quantization scheme are possible. Firstly, we could identify the time with the y direction. In this way, the system is a semi-infinite line with a boundary at $x = 0$,while the time runs form $-\infty$ to $+\infty$. The Hamiltonian as well as the Hilbert space of the theory depend explicitly on the boundary condition. The scattering properties of the theory are not determined only by the bulk S matrix, but also by the reflection at the boundary. Explicitly, we have a vacuum state $|0\rangle_B$, where the B remind us of the dependence from the boundary condition, a set of asymptotic states build by the action of the bulk creation operators $Z_i^\dagger(\theta)$ on $|0\rangle_B$ and also an elementary reflection amplitude

$$Z_a^\dagger(\theta)|0\rangle_B = R_a^b(\theta)Z_b^\dagger(-\theta)|0\rangle_B, \tag{2.93}$$

that describe the elastic bouncing of a particle with rapidity θ against the boundary.
Otherwise, we could turn our head and identify -x as the time. Therefore, the system is now an infinite line, the Hamiltonian as well as the Hilbert space coincide with the bulk ones, while the boundary in time appears as an initial state.
Let us adopt now the picture where y is the time. As we have previously emphasized, integrable field theories are characterized by an infinite set of local conserved quantities. So, it is natural to wonder what happens to these conserved quantities when we introduce a boundary in our system. In general, they are no more conserved *unless* we choose very carefully our boundary conditions. These very special boundary conditions have been extensively studied by Ghoshal and Zamolodchikov in [46]. Here we would like to summarize their findings.

- **Boundary Yang-Baxter Equations.** The reflection amplitude $R_a^b(\theta)$ satisfies

$$R_{a_2}^{c_2}(\theta_2)S_{a_1c_2}^{c_1d_2}(\theta_1+\theta_2)R_{c_1}^{d_1}(\theta_1)S_{d_2d_1}^{b_2b_1}(\theta_1-\theta_2) = \tag{2.94}$$
$$= S_{a_1a_2}^{c_1c_2}(\theta_1-\theta_2)R_{c_1}^{d_1}(\theta_1)S_{c_2d_1}^{d_2b_1}(\theta_1+\theta_2)R_{d_2}^{b_1}(\theta_2). \tag{2.95}$$

  If the system is in a infinite volume, the existence of infinite many (local) conserved quantities guarantees the factorizability of the scattering amplitude. Here, since these conservation laws are not spoiled by the presence of the boundary, we have that not only the scattering in the bulk is factorizable, but also the reflection at the boundary. This factorizability property is encoded in the boundary Yang Baxter equation (2.95) , whose graphical representation is shown in fig. 2.5.

- **Boundary Unitarity.** The unitarity condition for the reflection amplitude is

$$R_a^c(\theta)R_c^b(-\theta) = \delta_a^b, \tag{2.96}$$

  whose graphical interpretation is shown in fig. (2.5) . Notice that this

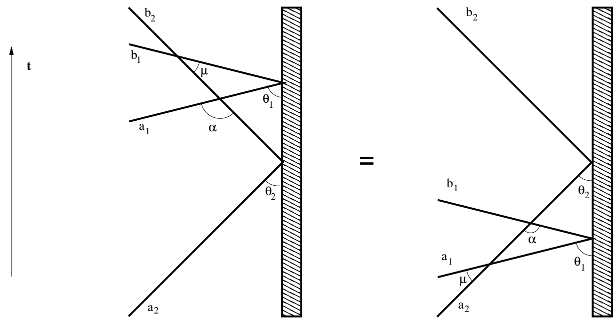

Figure 2.13: A graphical representation of the boundary Yang-Baxter equations. Here $\alpha = \theta_1 + \theta_2$ while $\mu = \theta_1 - \theta_2$.

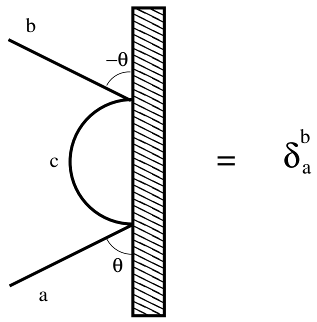

Figure 2.14: A pictorial representation of the boundary unitarity condition.

equation implies that the boundary scattering is purely reflective and no transmission occurs across the boundary.

- **Boundary Crossing.** The crossing properties of the reflection amplitude $R_a^b(\theta)$ can be expressed in terms of a boundary cross-unitarity condition. If we introduce the amplitude

$$K^{ab}(\theta) = R_{\bar{a}}^{b}\left(i\frac{\pi}{2} - \theta\right), \tag{2.97}$$

the boundary cross-unitarity condition reads as

$$K^{ab}(\theta) = S_{cd}^{ab}(2\theta) K^{dc}(-\theta). \tag{2.98}$$

- **Boundary Bootstrap.** Simple poles in the reflection amplitude can be interpreted as boundary bound states. Since we do not need them for our analysis, we refer the reader to the original literature [46] for further details on this topic.

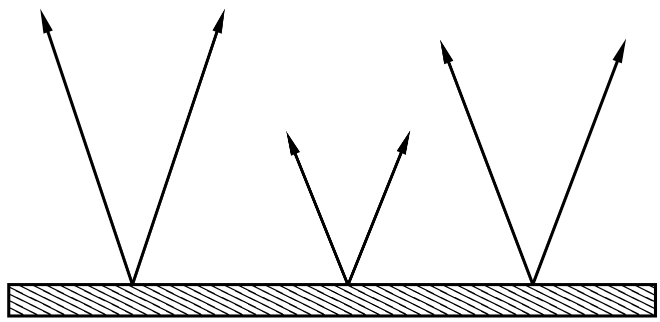

Figure 2.15: A graphical representation of an integrable boundary state (2.99) composed by pairs with opposite momentum.

As for the bulk S matrix, these conditions are a set of equation that can be used to determine the reflection amplitude R up the CDD ambiguity.
Let us now turn our head and consider $-x$ as the time. The boundary will appear now as a special initial state $|B\rangle$. It is quite natural to wonder if the boundary states $|B\rangle$ that correspond to integrable reflection amplitudes have a special form and what is their relation with $R_a^b(\theta)$. The answer is that, in absence of boundary bound states, $|B\rangle$ is simply an exponential of pairs with opposite rapidities ,

$$|B\rangle = \exp\left[\frac{1}{2}\int_{-\infty}^{+\infty} d\theta\, K^{ab}(\theta) Z_a^\dagger(-\theta) Z_b^\dagger(\theta)\right]|0\rangle, \tag{2.99}$$

Notice that the exponential in (2.99) is well defined, since

$$\left[K^{ab}(\theta) Z_a^\dagger(-\theta) Z_b^\dagger(\theta), K^{cd}(\theta) Z_c^\dagger(-\theta') Z_d^\dagger(\theta)\right], \tag{2.100}$$

thanks to the boundary Yang-Baxter equations (2.5).

Chapter 3

Quantum Quenches in Integrable Field Theories

In this chapter, we consider a quantum quench in an integrable field theory [1]. First of all, in sec. 3.1 , we define the problem we are interested in and we state our results. In the previous chapter, we have emphasized the power of the form factors program. However, in sec. 3.2 , we highlight some difficulties that emerge when we use form factors to compute thermal averages. It turns out that in the study of quantum quenches we face the very same problems, and our solution will be reminiscent of what is done in the thermal case. So, in sec. 3.3, we prove the results enunciated in 3.1, while in sec. 3.4 we show a simple example of the problem under investigation. Finally, in sec. 3.5, we discuss some recent and interesting papers [28, 47] and their relation with our work.

3.1 Definition of the Problem and Main Results

For simplicity, we consider an integrable quantum theory with only one kind of particle (e.g. the Sinh-Gordon model). So, the eigenstates are constructed by the action on the vacuum $|0\rangle$ (that we assume unique) of the Zamolodchikov-Fadeev algebra

$$\begin{aligned} &Z(\theta_1)\,Z(\theta_2) = S(\theta_1-\theta_2)\,Z(\theta_2)\,Z(\theta_1)\,,\\ &Z^\dagger(\theta_1)\,Z^\dagger(\theta_2) = S(\theta_1-\theta_2)\,Z^\dagger(\theta_2)\,Z^\dagger(\theta_1)\,,\\ &Z(\theta_1)\,Z^\dagger(\theta_2) = S(\theta_2-\theta_1)\,Z^\dagger(\theta_2)\,Z(\theta_1) + 2\,\pi\,\delta(\theta_1-\theta_2)\,. \end{aligned} \tag{3.1}$$

However, our considerations are quite general: they hold for any model (that satisfies the previous assumptions) since they do not rely on the specific

form of the S matrix. This fact strongly suggest that these results could be trivially generalized to model with more than one particle, at least for diagonal scattering theories.
In order to describe a quench problem, we should choose an initial state. We opt for a BCS-like state $|B\rangle$

$$|B\rangle = \exp\left[\frac{1}{2}\int_{-\infty}^{+\infty}\frac{d\theta}{2\,\pi}G(\theta)Z^{\dagger}(-\theta)\,Z^{\dagger}(\theta))\right]|0\rangle, \tag{3.2}$$

that is the exponential coherent state of pairs with opposite rapidities. There are two reasons behind this choice. First of all, we have seen (sec. 1.2.2) that such states describe the simplest possible quench process, a mass quench in a free theory. Moreover, these states emerge naturally in the contest of integrable boundary conditions (sec. 2.5). A quantum quench can be seen as a statistical physics problem in a confined geometry (1.2.1), where the initial state plays the role of a boundary condition. Therefore, choosing as the initial state an integrable one means that we are considering the most integrable situation: the Hamiltonian in the bulk is integrable as well as the boundary condition. However, as we have emphasized in sec. 2.5 , integrability implies that the pair amplitude $G(\theta)$ must satisfy a set of conditions: the boundary Yang-Baxter condition (2.95), the boundary unitarity condition (2.96) and the boundary cross-unitarity condition (2.98). Indeed, the Yang-Baxter condition is automatically satisfied if we are dealing with only one particle. Similarly, the cross-unitarity condition

$$G(\theta) = S(2\theta)G(-\theta) \tag{3.3}$$

is simply a consequence of the invariance of the integral $\int d\theta G(\theta)Z^{\dagger}(-\theta)\,Z^{\dagger}(\theta)$ under the change of variable $\theta \to -\theta$. Indeed, any function $F(\theta)$ can be decomposed as

$$F(\theta) = G(\theta) + H(\theta) = \frac{F(\theta) + S(2\theta)F(-\theta)}{2} + \frac{F(\theta) - S(2\theta)F(-\theta)}{2}, \tag{3.4}$$

where $G(\theta)$ and $H(\theta)$ satisfies

$$G(\theta) = S(2\theta)G(-\theta), \qquad H(\theta) = -S(2\theta)H(-\theta). \tag{3.5}$$

However,

$$\int d\theta H(\theta)Z^{\dagger}(-\theta)\,Z^{\dagger}(\theta) = \int d\theta H(-\theta)Z^{\dagger}(\theta)\,Z^{\dagger}(-\theta) = \\ \int d\theta H(-\theta)S(2\theta)Z^{\dagger}(-\theta)\,Z^{\dagger}(\theta) = -\int d\theta H(\theta)Z^{\dagger}(-\theta)\,Z^{\dagger}(\theta), \tag{3.6}$$

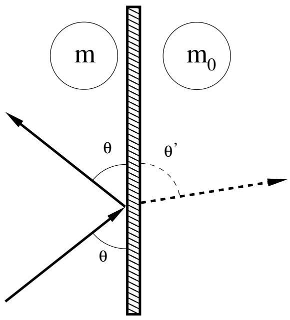

Figure 3.1: The boundary conditon for a mass quench allows for transmission. Notice that,for free theories, there is no particle production at the boundary and the transmitted particle has always the same momentum of the incoming one, since momentum is conserved at the boundary. However, since the mass is different at $x < 0$ and $x > 0$, the rapidity changes from θ to θ'.

hence $\int d\theta H(\theta) Z^{\dagger}(-\theta)\, Z^{\dagger}(\theta) = 0$. In the following we will see that the unitary condition plays no role, and therefore we don't have to assume it. As a matter of facts, it is indeed violated for the mass quench in a free fermionic theory, a signal that the boundary is not transmissionless (fig. 3.1).

However, it is important that the $G(\theta)$ vanishes for large $|\theta|$. Physically, this is equivalent to ask that we do not excite modes with arbitrarily high energy and therefore the energy density (the energy per unit of volume) of the initial state is finite. This is usually not the case for integrable amplitudes $K(\theta)$, since they go to a non vanishing constant for large $|\theta|$. Therefore, a regularizator is needed, as for example the extrapolation length used in the conformal case (1.2.1).

Given an initial state as (3.2), we would like to study the thermalization properties of local or semilocal operators $\mathcal{O}(x)$. Therefore, the matrix elements of these operators satisfy the axioms of the form factors program (sec. 2.4.4). We emphasize that, even in the following we will speak about local operators for the sake of brevity, our analysis is valid also for semilocal operators as, for example, the order parameter of the Ising model.

Our statement is that the long time limit the expectation value of a local operators over the initial state (3.2) is described by a generalized Gibbs ensemble,

$$\hat{\rho}_{\lambda} = \frac{\exp\left(-\int d\theta \lambda(\theta)\, \hat{n}(\theta)\right)}{Z}, \tag{3.7}$$

where $\hat{n}(\theta) = Z^{\dagger}(\theta) Z(\theta)$ and $\lambda(\theta)$ is a Lagrange multiplier fixed by the initial state. More precisely, if we define

$$\overline{\mathcal{O}} = \lim_{t\to+\infty} \langle \mathcal{O}(x,t)\rangle_B = \lim_{t\to+\infty} \frac{\langle B|\mathcal{O}(x,t)|B\rangle}{\langle B|B\rangle}, \tag{3.8}$$

that is clearly position-independent due to the translational invariance of (3.2), we have that

$$\overline{\mathcal{O}} = \sum_{n=0}^{+\infty} \frac{1}{n!} \int \prod_{i=1}^{n} \frac{d\theta_i}{(2\pi)} \left[\frac{|\tilde{G}(\theta_i)|^2}{1 - S(0)\,|\tilde{G}(\theta_i)|^2} \right] \langle \theta_n \dots \theta_1 | O(0) | \theta_1 \dots \theta_n \rangle_{conn} \,. \tag{3.9}$$

Here $|\tilde{G}(\theta)|^2$ satisfy the TBA-like equation

$$|\tilde{G}(\theta)|^2 = |G(\theta)|^2 \, \exp\left[\int \frac{d\theta'}{2\,\pi}\, \varphi(\theta-\theta') \log[1 + |\tilde{G}(\theta)|^2] \right] \,, \tag{3.10}$$

where φ is the derivative of the phase shift

$$\varphi(\theta) = -i \frac{d}{d\theta} \log(S(\theta)), \tag{3.11}$$

while $\langle \theta_n \dots \theta_1 | O(0) | \theta_1 \dots \theta_n \rangle_{conn}$ are the connected form factors, a properly regularized expression for the matrix elements (3.18).

3.2 Kinematical Singularities and the LeClair-Mussardo Formula

In sec. 2.4.4 we have explained the basic ideas of the form factors program. Indeed, in an integrable field theory, we can compute any matrix element of local (or semi-local) operator through the form factors

$$F_n^{\mathcal{O}}(\theta_1, \dots, \theta_n) = \langle 0 | \mathcal{O}(0,0) | \theta_1, \dots, \theta_n \rangle, \tag{3.12}$$

that satisfy a set of consistency equations. However, in the previous chapter, we have left open one point: how to derive the most generic matrix element $\langle \theta'_m \dots \theta'_1 | \mathcal{O}(0,0) | \theta_1 \dots \theta_n \rangle$from the form factors. The reason is that this issue is quite subtle and it doesn't have a completely satisfactory answer. We have already seen that, if $\theta'_i \neq \theta_i \, \forall i,j$, then

$$\langle \theta_n \dots \theta_1 | \mathcal{O}(0,0) | \theta'_1 \dots \theta'_m \rangle = F^{\mathcal{O}}_{n+m}(\theta'_1, \dots, \theta'_m, \theta_1 - i\pi, \theta_n - i\pi). \tag{3.13}$$

However, if $\theta'_i = \theta_j$ for some i,j, then we have to take into account also the disconnected contributions. Phenomenological reasoning in the infinite volume suggest the following formula: if we denote with A and B two sets of rapidities, then

$$\langle A|\mathcal{O}(0)|B\rangle = \sum_{A=A_1\cup A_2\,;\,B=B_1\cup B_2} S_{AA_1} S_{BB_1} \langle A_1^+|\mathcal{O}(0)|B_1\rangle\langle A_2|B_2\rangle, \qquad (3.14)$$

where the sum is over all the possible ways of splitting the sets A/B in two subsets $A_1/\ B_1$ and A_2/B_2 while S_{AA_1} and S_{BB_1} are the products of $S(\theta)$ we need to rearrange the rapidities in the proper order, namely

$$\langle A| = S_{AA_1}\langle A_2 A_1|\,, \qquad (3.15)$$

$$|B\rangle = S_{BB_1}|B_1 B_2\rangle\,. \qquad (3.16)$$

The symbol A_1^+ in (3.14) denotes that each rapidity $\theta_1\ldots\theta_r$ in A_1 is shifted by an infinitesimal imaginary amount $i\epsilon_i$ so that $\langle A_1^+|\mathcal{O}|B_1\rangle$ is simply related to the corresponding form factor by

$$\langle A_1^+|\mathcal{O}(0)|B_1\rangle = \langle 0|\mathcal{O}(0)|B_1 A_1^+ - i\,\pi\rangle\,. \qquad (3.17)$$

When the ϵ_i are finite, the form factors are (for real rapidities) regular functions. However, at the end of the day, we would like to take the limit $\epsilon_i \to 0$: in this limit the form factors usually diverge. The simplest case of this circumstance is provided by the 2-particle matrix element $\langle\theta|\mu|\theta'\rangle$ where μ is the disorder operator of the Ising model (2.86), which indeed diverges when $\theta = \theta'$. This discussion shows that a prescription is needed for handling these kinematical divergencies. The one proposed in [48,49] consists of taking only the *regular* part of (3.17) and discarding *all* the terms proportional to an inverse power of ϵ_i

$$\begin{aligned}&\langle\theta_n,\ldots,\theta_1|\mathcal{O}(0)|\theta'_1,\ldots,\theta'_m\rangle_{conn} = \qquad (3.18)\\ &= \text{Finite Parts}\left[\lim_{\epsilon_i\to 0}\langle 0|\mathcal{O}(0)|\theta'_1,\ldots,\theta'_m,\theta_n - i\,\pi + i\epsilon_n,\ldots,\theta_1 - i\,\pi + i\epsilon_1\right].\end{aligned}$$

It should be stressed, however, that this prescription alone is not enough to properly take care of all the divergencies and, usually, it must be supplemented with extra corrective factors coming from the Bethe-ansatz technique [48]. For example, let us consider as a density matrix of the system

$$\hat{\rho}_\lambda = \frac{\exp\left(-\int d\theta\lambda(\theta)\,\hat{n}(\theta)\right)}{Z}, \qquad (3.19)$$

where $\hat{n}(\theta) = Z^\dagger(\theta)Z(\theta)$ and $\lambda(\theta)$ is an appropriate function of θ. Notice that, if $\lambda(\theta) = \frac{1}{T}\,m\,\cosh(\theta)$ the density matrix (3.19) describes the familiar canonical ensemble, while, in the more general case, it can be associated to the generalized Gibbs ensemble (1.9). Obviously, it would be nice to write down the average of a local operator $\mathcal{O}$ w.r.t. the density matrix (3.19) in term of these form factors. But, if we do it applying blindly the prescription (3.18) we end up with the result

$$\langle O\rangle_{\hat\rho} = Tr\,(\hat\rho\, O(0)) = \qquad\qquad !\,\textbf{WRONG}\,! \tag{3.20}$$
$$= \sum_{n=0}^{+\infty} \frac{1}{n!}\int \frac{d\theta_1\ldots d\theta_n}{(2\pi)^n}\prod_{i=1}^{n}\left[\frac{e^{-\lambda(\theta_i)}}{1-S(0)\,e^{-\lambda(\theta_i)}}\right]\langle\theta_n,\ldots,\theta_1|O(0)|\theta_1,\ldots,\theta_n\rangle_{conn}\,,$$

which is wrong since it does not agree with the thermodynamic Bethe ansatz. It was firstly conjectured by LeClair and Mussardo [48] that the correct expression is instead

$$\langle O\rangle_{\hat\rho} = Tr\,(\hat\rho\, O(0)) = \tag{3.21}$$
$$= \sum_{n=0}^{+\infty} \frac{1}{n!}\int \frac{d\theta_1\ldots d\theta_n}{(2\pi)^n}\prod_{i=1}^{n}\left[\frac{e^{-\tilde\lambda(\theta_i)}}{1-S(0)\,e^{-\tilde\lambda(\theta_i)}}\right]\langle\theta_n,\ldots,\theta_1|O(0)|\theta_1,\ldots,\theta_n\rangle_{conn}\,,$$

where the $\tilde\lambda$ are dressed according to the integral equation

$$\tilde\lambda(\theta) = \lambda(\theta) - \int \frac{d\theta'}{2\,\pi}\,\varphi(\theta-\theta')\log[1+e^{-\tilde\lambda(\theta')}], \tag{3.22}$$

φ being the derivative of the phase shift (3.11).
Originally, (3.21) was only an educated guess but its correctness was confirmed by subsequent checks [50–52]. An interesting approach to this formula has been put forward by Pozsgay and Takacs [53–55], who have considered a system in a finite volume, thus discretizing the rapidities to regularize the kinematical singularities, and taking the infinite volume limit only at the end of the calculations. This set of ideas has finally lead to a satisfactory proof by Balázs Pozsgay of the LeClair-Mussardo formula (3.21) [47]-see sec. 3.5 for more details. However, a complete and satisfactory way to deal with the divergencies that arise in (3.14) is still lacking. This is reflected by the fact that it is still unknown if the LeClair-Mussardo formula -or a suitable generalization-holds for two points functions, even if there have been recent progresses in this area [56, 57].

3.3 A Proof of the Generalized Gibbs Ensemble

In this section, we would like to prove our statement (3.9). For the sake of clarity, we will divide our proof in two parts: in sec. 3.3.1 we discuss the

general ideas used in this proof, while the combinatorial technicalities are shown in sec. 3.3.2.
We would like to emphasize that, as it will be quite clear from our exposition, our arguments cannot be considered at all a rigorous proof of our statement, due to the difficulties in the treatment of the divergencies discussed in the previous section. However, we believe that our arguments are quite convincing, and they have been supported by recent developments in the field, as we discuss in sec. 3.5.

3.3.1 Main Ideas of Our Proof

In this section, we show that $\overline{\mathcal{O}}$ (3.8) can be expressed as an average over a density matrix like (3.7). At first sight, it seems very unlikely that we can do so: our boundary state (3.2) has a very peculiar structure, since it is the superposition of pairs of opposite rapidity, while in an average over an ensemble (3.7) there is no sign of such a structure. How comes that the system retains no memory of this pair structure in the long time limit?

First of all, a trivial remark: since $|B\rangle$ is translational invariant, $\langle O(x,t)\rangle_B$ do not depend on x and therefore, from now on, we will set $x = 0$. In principle, it is quite clear what we have to do in order to compute (3.8): first we expand the exponential in (3.2) and, taking into account that the Hamiltonian is diagonal in the particle basis, we thus arrive to the double sum

$$\langle O(x,t)\rangle_B = \frac{1}{\langle B|B\rangle}\sum_{n,l=0}^{+\infty}\frac{1}{n!\,l!}\int\frac{d\theta_1\dots d\theta_n}{(4\,\pi)^n}\,\frac{d\theta'_1\dots d\theta'_l}{(4\,\pi)^l}e^{2\,i\,t(E_n(\theta)-E_l(\theta'))}$$
$$\left[\prod_{i=1}^{n}\overline{G}(\theta_i)\right]\left[\prod_{j=1}^{l}G(\theta'_j)\right]\langle\theta_n,-\theta_n,\dots\theta_1,-\theta_1|\mathcal{O}|-\theta'_1,\theta'_1,\dots-\theta'_l,\theta'_l\rangle\,,\quad(3.23)$$

where we used the short-hand

$$E_n(\theta) = m\sum_{i=1}^{n}\cosh(\theta_i)\,. \quad (3.24)$$

However it is difficult to compute the long time limit directly from (3.23). The reason is that the matrix element in (3.23) are not regular functions (rather they have delta-like contributions) and, in such a case, we cannot apply a stationary phase argument. In order to isolate the singular parts, we have to employ eqn.(3.14). Consider, for instance, the term with $n = l = 1$

in the numerator of (3.23), i.e.

$$\int \frac{d\theta_1}{4\pi}\frac{d\theta_1'}{4\pi}\, e^{2it(E_1(\theta)-E_1(\theta'))}\overline{G}(\theta_1)G(\theta_1')\langle\theta_1,-\theta_1|\mathcal{O}|-\theta_1',\theta_1'\rangle\,. \tag{3.25}$$

Applying the crossing relation (3.14), we can recast this term as

$$\begin{aligned}&\int \frac{d\theta_1}{4\pi}\frac{d\theta_1'}{4\pi}\, e^{2it(E_1(\theta)-E_1(\theta'))}\;\overline{G}(\theta_1)G(\theta_1')\langle\theta_1^+,-\theta_1^+|\mathcal{O}|-\theta_1',\theta_1'\rangle + \\ &+\int \frac{d\theta_1}{2\pi}|G(\theta_1)|^2\langle\theta_1^+|\mathcal{O}|\theta_1\rangle + \langle 0|\mathcal{O}|0\rangle\int \frac{d\theta_1}{4\pi}\frac{d\theta_1'}{4\pi}\,\overline{G}(\theta_1)G(\theta_1')\langle\theta_1,-\theta_1|-\theta_1',\theta_1'\rangle\,,\end{aligned} \tag{3.26}$$

where we make use of the symmetry properties (3.3) of G. Actually, the inner product $\langle\theta_1,-\theta_1|-\theta_1',\theta_1'\rangle$ above is divergent in the infinite volume limit and it should be regularized by putting the system in a box of length L, obtaining

$$(\delta(\theta-\theta'))^2 = \frac{m\,L}{2\,\pi}\cosh(\theta)\delta(\theta-\theta'). \tag{3.27}$$

This, however, is not an important contribution since, as shown in sec. 3.3.2, all these inner products cancel out with the denominator of (3.23). What is really crucial is that, apart from these infinite volume divergencies that we can easily regularize, the integrands in (3.26) are all well-behaved functions: hence we can now easily take the infinite time limit $t \to +\infty$, so that (3.26) simply becomes

$$\int \frac{d\theta_1}{2\pi}|G(\theta_1)|^2\langle\theta_1^+|\mathcal{O}|\theta_1\rangle + \langle 0|\mathcal{O}|0\rangle\int \frac{d\theta_1}{4\pi}\frac{d\theta_1'}{4\pi}\,\overline{G}(\theta_1)G(\theta_1')\langle\theta_1,-\theta_1|-\theta_1',\theta_1'\rangle \tag{3.28}$$

because the first term in (3.26) vanishes for the fast oscillation of its integrand.

In the light of this example, the strategy to compute the expectation values of local operators can be stated as follows.

1. We first expand the exponential in the numerator of (3.8), ending up with the double sum (3.23).

2. Then, we use (3.14) in order to isolate the delta-like terms and, after having done that, we take the infinite time limit, where all terms that explicitly depend on time go to zero, due to the fast oscillation of the integrand. This is a simple consequence of the stationary phase argument, that can also be seen in the following way. If the infinite time limit exists (and the stationary phase argument assures us that it does exist), then it must coincide with the temporal average

$$\overline{\mathcal{O}} = \lim_{t\to+\infty}\langle\mathcal{O}(x,t)\rangle_B = \lim_{T\to+\infty}\frac{1}{T}\int_0^T dt'\langle O(x,t')\rangle_B\;. \tag{3.29}$$

We know that the time-dependent part of the numerator of (3.23) consists of a sum of terms as

$$\int d\theta_1 \ldots d\theta_n \, d\theta'_1 \ldots d\theta'_l e^{2\,i\,t(E_n(\theta)-E_l(\theta'))} F(\theta_1 \ldots \theta_n, \theta'_1 \ldots \theta'_l) \,, \qquad (3.30)$$

where F is the regular function obtained by applying (3.14). So, the only contributions to the infinite time limit comes from the region $E_n(\theta) = E_l(\theta')$, whose Lebesgue measure is zero, so the integral goes to zero since F has no delta-like term.

With these steps in mind, it is a nice combinatorial exercise (see sec. 3.3.2) to show that $\overline{\mathcal{O}}$ can be finally expressed as

$$\overline{\mathcal{O}} = \sum_{n=0}^{+\infty} \frac{1}{n!} \int \prod_{i=1}^{n} \frac{d\theta_i}{(2\pi)} \left[\frac{|G(\theta_i)|^2}{1 - S(0)\,|G(\theta_i)|^2} \right] \langle \theta_n + i\,\epsilon_n \ldots \theta_1 + i\,\epsilon_1 | O(0) | \theta_1 \ldots \theta_n \rangle \,. \qquad (3.31)$$

However, (3.31) is still a meaningless expression, since we have to regularize it in a proper way. One way to do it is by analogy with LeClair and Mussardo formula, discussed in section 3.2. When we perform an average over a density matrix (3.19), we end up with the following expression

$$\overline{\mathcal{O}} = \sum_{n=0}^{+\infty} \frac{1}{n!} \int \prod_{i=1}^{n} \frac{d\theta_i}{(2\pi)} \left[\frac{e^{-\lambda(\theta)}}{1 - S(0)\,e^{-\lambda(\theta)}} \right] \langle \theta_n + i\,\epsilon_n \ldots \theta_1 + i\,\epsilon_1 | O(0) | \theta_1 \ldots \theta_n \rangle \,. \qquad (3.32)$$

LeClair and Mussardo suggested that the proper way to regularize the $\epsilon_i \to 0$ limit of this expression is to take the the connected part of the form factors (3.18) and to dress $\lambda(\theta)$ according to the integral equation (3.22). This regularization scheme holds for every function $\lambda(\theta)$. The situation is the same in equation (3.31), with $|G(\theta)|^2$ that plays the role of $e^{-\lambda(\theta)}$. So, the natural way of regularize (3.31) lead us to

$$\overline{\mathcal{O}} = \sum_{n=0}^{+\infty} \frac{1}{n!} \int \prod_{i=1}^{n} \frac{d\theta_i}{(2\pi)} \left[\frac{|\tilde{G}(\theta_i)|^2}{1 - S(0)\,|\tilde{G}(\theta_i)|^2} \right] \langle \theta_n \ldots \theta_1 | O(0) | \theta_1 \ldots \theta_n \rangle_{conn} \,, \qquad (3.33)$$

where $|\tilde{G(\theta)}|^2$ is dressed in the same way as the term $e^{-\tilde{\lambda}(\theta)}$ entering the thermodynamic Bethe ansatz

$$|\tilde{G}(\theta)|^2 = |G(\theta)|^2 \, \exp\left[\int \frac{d\theta'}{2\,\pi} \, \varphi(\theta - \theta') \log[1 + |\tilde{G}(\theta)|^2] \right] \,. \qquad (3.34)$$

The above dressing formula is based on the LeClair and Mussardo conjecture (it has actually the same mathematical structure) and on the possibility to exchange the $\epsilon_i \to 0$ and $t \to +\infty$ limits. Assuming that such a regularization scheme is indeed correct, it turns out that that the long time limit of $\langle \mathcal{O}(x,t)\rangle_B$ could be described by a generalized Gibbs ensemble (3.19), where the constant of motions are simply given by the occupation number $\hat{n}(\theta)$ and the function $\lambda(\theta)$ is fixed by the conditions

$$\frac{\langle B|\hat{n}(\theta)|B\rangle}{\langle B|B\rangle} = Tr\left(\hat{\rho}_\lambda \, \hat{n}(\theta)\right), \tag{3.35}$$

thus proving Rigol *et al.*'s conjecture for integrable field theory. If we look at our starting point, this result is quite unexpected: when we expanded the exponential in (3.8) we had a double sum, and it was only thanks to the infinite time limit that we could rewrite it as a single summation. Moreover, while the boundary state (3.2) is formed by pairs of particles with opposite rapidity, this feature is completely lost in the final expression (3.33).

3.3.2 Combinatorial Details of Our Proof

In this subsection we would like to clarify the combinatorial details omitted in the previous subsection. Therefore, we will show that, for $t \to +\infty$,

$$\begin{aligned} &\frac{\langle B|\mathcal{O}(x,t)|B\rangle}{\langle B|B\rangle} \to \\ &\to \sum_{n=0}^{+\infty} \frac{1}{n!} \int \frac{d\theta_1 \ldots d\theta_n}{(2\pi)^n} \prod_{i=1}^{n} \left[\frac{|G(\theta_i)|^2}{1 - S(0)\,|G(\theta_i)|^2}\right] \langle \theta_n^+ \ldots \theta_1^+ |O(0)|\theta_1 \ldots \theta_n\rangle\,, \end{aligned} \tag{3.36}$$

where $\theta_i^+ = \theta_i + i\epsilon_i$. In this section, we will call the form factors like $\langle \theta_n^+ \ldots \theta_1^+|O(0)|\theta_1 \ldots \theta_m\rangle$ *regular* form factors, in order to distinguish them from the *complete* form factors $\langle \theta_n \ldots \theta_1|O(0)|\theta_1 \ldots \theta_m\rangle$, that have also delta-like contributions. For ϵ_i finite these regular form factors are continuous functions but, despite their name, they can have a singular $\epsilon_i \to 0$ limit, hence the need of the regularization procedure previously discussed. Let's firstly briefly summarize the main steps of the proof: after we expand the numerator of the l.h.s. of (3.36) (as done in (3.23)), we obtain

$$\begin{aligned} \langle B|\mathcal{O}(x,t)|B\rangle = &\sum_{n,l=0}^{+\infty} \frac{1}{n!\,l!} \int \frac{d\theta_1 \ldots d\theta_n}{(4\,\pi)^n} \frac{d\theta'_1 \ldots d\theta'_l}{(4\,\pi)^l} e^{2\,i\,t(E_n(\theta) - E_l(\theta'))} \cdot \\ &\cdot \left[\prod_{i=1}^{n} \overline{G}(\theta_i)\right] \left[\prod_{j=1}^{l} G(\theta'_j)\right] \langle \theta_n, -\theta_n, \ldots \theta_1, -\theta_1|\mathcal{O}| - \theta'_1, \theta'_1, \ldots - \theta'_l, \theta'_l\rangle\,. \end{aligned} \tag{3.37}$$

Then, applying repeatedly the crossing relation (3.14) to the matrix elements in (3.37), we arrive to an expression in which we can take the infinite time limit. In this limit, (3.37) reduces to the r.h.s. of (3.36) times the denominator of the l.h.s

$$\mathcal{Z} = \langle B|B\rangle = \langle B|e^{iHt}e^{-iHt}|B\rangle = \sum_{n=0}^{+\infty} \mathcal{Z}_n\,, \tag{3.38}$$

where

$$\mathcal{Z}_n = \frac{1}{n!^2}\int \frac{d\theta_1\dots d\theta_n}{(4\,\pi)^n}\,\frac{d\theta'_1\dots d\theta'_n}{(4\,\pi)^n}\left[\prod_{i=1}^{n}\overline{G}(\theta_i)G(\theta'_i)\right]\cdot \tag{3.39}$$
$$\cdot\exp(2\,i\,t\,(E_n(\theta)-E_n(\theta'))\langle\theta_n,-\theta_n,\dots\theta_1,-\theta_1|-\theta'_1,\theta'_1,\dots-\theta'_n,\theta'_n\rangle\,,$$

thus completing the proof.

Introductory remarks. Here we would like to highlight some basic facts that we will find useful for our proof. First of all, we notice that, when we recast (3.37) in terms of the regular form factors, the only terms that survive in the infinite time limit are the time-independent ones. As a consequence, we can discard all the terms in the double sum in (3.37) where $n \neq l$. Moreover, from (3.14) it is clear that the term $\langle 0|\mathcal{O}|0\rangle$ in the r.h.s. of (3.36) is correct, so in the following we will not consider this contribution.

In order to follow more easily our ideas, it may be useful to develop a diagrammatic representation. In fig. 3.2 it is shown a *bra* state with n pairs (a *ket* state can be introduced in a similar way). An inner product between a *bra* and a *ket* both made of n pairs can be represented as the sum of all the possible ways to link together the particles of the ket with the particles of the bra, each link meaning a contraction between the corresponding particles. Of course, one should be careful about the permutation of particles and the corresponding S matrix, but we will take care later of these details. We are interested in the matrix elements of the operator $\mathcal{O}$ between states made of Cooper pairs. In particular, our aim is to reduce the full matrix element $\langle\theta_n,-\theta_n,\dots\theta_1,-\theta_1|\mathcal{O}|-\theta'_1,\theta'_1,\dots-\theta'_n,\theta'_n\rangle$ to the regular form factors $\langle\theta^+_{i_r},\dots,\theta^+_{i_1}|\mathcal{O}|\theta'_{j_1},\dots,\theta'_{j_r}\rangle$. These regular terms can be diagrammatically represented as in fig. 3.2, where the operator $\mathcal{O}$ has $2r$ legs: r of them are connected to the particles in the bra, while the other r are connected to the particles in the ket. So, our combinatorial problem reduces to a problem in which we have to connect the legs of the operator to the bra/ket and the remaining particles together, according to (3.14). In order to get the right combinatorial coefficients, we have to remember what follows.

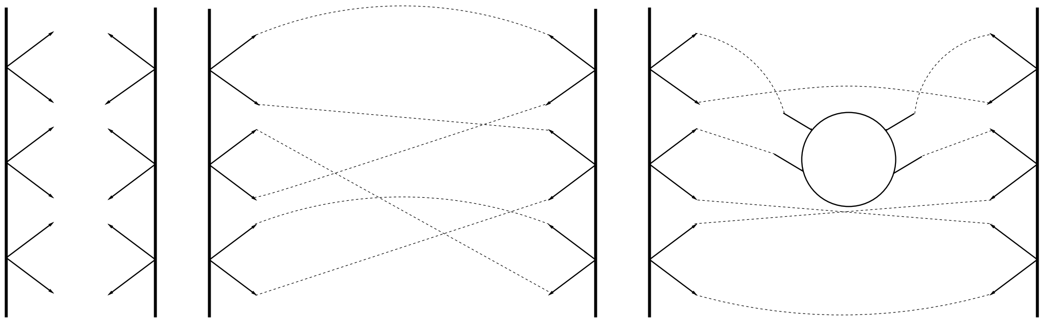

Figure 3.2: In this figure we present the building blocks of our diagrammatic representation. On the left, we can see a bra and a ket with $n = 3$ Cooper pairs. In the middle, we see one of the diagrams that represent the inner product of the bra and the ket. The figure on the right shows a term proportional to the regular form factor $\langle\theta_2^+,\theta_1^+|\mathcal{O}|\theta_1',\theta_2'\rangle$. The circle in the middle stands for the operator $\mathcal{O}$, that has $2r = 4$ legs, half connected to the bra and half to the ket.

- As we stated before, only the matrix elements with the same numbers of particles in the ket and in the bra give a non vanishing contribution in the long times limit. Therefore, the number of legs connected to the bra is always equal to the number of legs connected to the ket and, from now on, we will only specify the number of particles connected to the bra.
- In principle, when we consider the matrix element between two states with n pairs, i. e. $\langle\theta_n,-\theta_n,\dots\theta_1,-\theta_1|\mathcal{O}|-\theta_1',\theta_1',\dots-\theta_n',\theta_n'\rangle$, we could expect to end up with a sum of regular terms with r legs linked to the bra, with $r \leq 2n$. However, it turns out that $r \leq n$: if we connect r particles to the operator, we are left with $2n-r$ delta functions, and to suppress all the time dependencies, we have to eliminate n integration variables, hence $r \leq n$.
- Finally, if we link a particle to the operator, its pair partner *cannot* be connected to O, otherwise their time dependence survives.

The disconnected terms. Let us show now how some contributions to (3.36) (the *disconnected terms*) cancel out with the denominator $\mathcal{Z}$ (3.38). In order to understand this point, we analyze the matrix elements $\langle\theta_2,-\theta_2,\theta_1,-\theta_1|\mathcal{O}|-\theta_1',\theta_1',-\theta_2',\theta_2'\rangle$. As it is shown in fig. 3.3, we have essentially two types of diagram. Let's focus our attention on the second one: it is clear that the contribution in the dotted box factorize from the integral with the form factors. Hence, if we have a set of particles that is completely

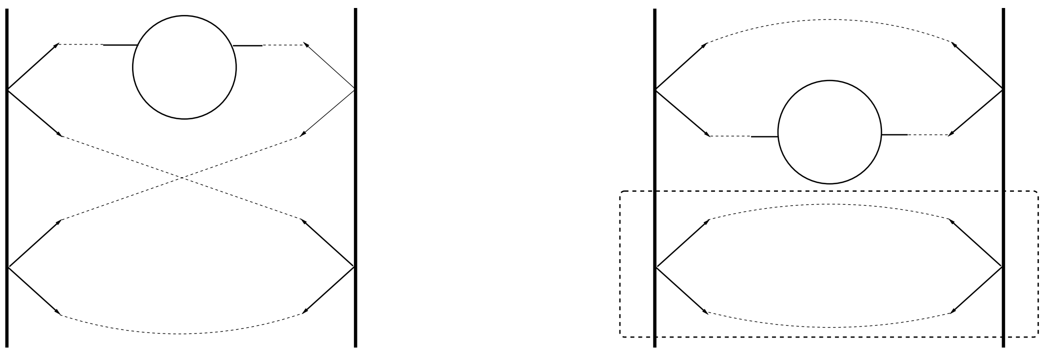

Figure 3.3: Two different diagrams: in the left one all the particles are connected to the operator, while in the right one there is a contribution that factorizes.

disconnected from the operator $\mathcal{O}$ (this means that no one of these particles or their partners is connected to $\mathcal{O}$ or to a particle whose partner is connected to $\mathcal{O}$), its contribution factorizes.

What we want to show now is that these disconnected pieces cancel out with the denominator $\mathcal{Z}$. Let us consider the term in (3.37) with n pairs in the bra (let's denote it as $\mathcal{O}_n$), and let's focus our attention on the contribution $\mathcal{O}_{[n,k]}$, such that k pairs in the bra (as well as k pairs in the ket) are disconnected and so only $n-k$ pairs in the bra are connected to $\mathcal{O}$. Of course, a similar term can also be obtained from $\mathcal{O}_{n-k}$, when no particle is disconnected. With our notation, this term can be written as $\mathcal{O}_{[n-k,0]}$. If we are able to show that, for any n and $k \leq n$,

$$\mathcal{O}_{[n,k]} = \mathcal{O}_{[n-k,0]}\, \mathcal{Z}_k \,, \tag{3.40}$$

then it follows that, when we sum over all n and $k \leq n$, (3.37) becomes

$$\left(\sum_{m=0}^{+\infty} \mathcal{O}_{[m,0]} \right) \mathcal{Z} \,, \tag{3.41}$$

hence the disconnected pieces cancel out with $\mathcal{Z}$.

The proof of (3.40) is actually quite simple. It is clear that the inner product of the k disconnected pairs gives an integral proportional to $\mathcal{Z}_k$, as well as $\mathcal{O}_{[n,k]}$ is clearly proportional to $\mathcal{O}_{[n-k,0]}$. So, in order to complete the proof, we have only to check the proportionality constant. $\mathcal{O}_{[n,k]}$ has an overall coefficient $\frac{1}{n!^2}$ from the expansion of the exponentials, while we can choose the k disconnected pairs in $\left(\frac{n!}{k!(n-k)!}\right)^2$ equivalent ways, since the creation operator of a pair commutes ($[Z^\dagger(-\theta)\, Z^\dagger(\theta), Z^\dagger(-\theta')\, Z^\dagger(\theta')] = 0$).

Nicely, $\frac{1}{(n-k)!^2}$ is the correct coefficient for $\mathcal{O}_{[n-k,0]}$ and $\frac{1}{k!^2}$ is the right one for $\mathcal{Z}_k$, thus concluding the proof of 3.40.

The last step. Let us conclude our proof, showing that $\mathcal{O}_{[m,0]}$ has actually the right structure to give (3.36). Let's call $\mathcal{O}_{\{m,r\}}$ the contribution where the operator has r legs connected to the bra, the bra has m pairs and no particle is disconnected from $\mathcal{O}$. We have already pointed out that for $t \to +\infty$

$$\mathcal{O}_{[m,0]} = \sum_{r=0}^{m} \mathcal{O}_{\{m,r\}}, \tag{3.42}$$

since the sum is restricted to $r \leq m$. So, in order to get (3.36), we need only to show that

$$\mathcal{O}_{\{m,r\}} = \frac{1}{r!} \int \frac{d\theta_1 \dots d\theta_r}{(2\pi)^r} \langle \theta_r^+ \dots \theta_1^+ | O(0) | \theta_1 \dots \theta_r \rangle \cdot \tag{3.43}$$
$$\cdot \sum_{i_1,\dots,i_r}{}' \left[S(0)^{i_1-1} (|G(\theta_1)|^2)^{i_1} \dots S(0)^{i_r-1} (|G(\theta_r)|^2)^{i_r} \right],$$

where the summation $\sum_{i_1,\dots,i_r}{}'$ is over all the positive integers i_j such that $\sum_j i_j = m$.

In order to prove (3.43), we need to be a little careful with the ordering of the particles and the labeling of the rapidities. However, if we exchange two particles, the contribution is same, since (as we already know) the pairs do commute while the exchange of two particles forming a pair is equivalent to change of the integrable variable $\theta \to -\theta$. This is a consequence of the symmetry of G (3.3).

Before concluding our proof of (3.36), we need to understand how to label the rapidity. We start with $2m$ integration variables and the delta-functions reduce them to r. So, we use the convention to label the rapidities as in (3.43): we call θ_1 the rapidity of the particle of the bra closest to $\mathcal{O}$ and we take advantage of delta functions in such a way that the rapidity of the particle in the ket nearest to $\mathcal{O}$ is also θ_1, and so on.

We can now finally show that we obtain exactly the structure (3.43). First of all, from our previous reasoning about the long time limit, it is clear that, performing all the possible contractions, for the term in the second row of (3.43) we arrive to an expression as

$$\sum_{i_1,\dots,i_r}{}' \left[c_1(i_1) (|G(\theta_1)|^2)^{i_1} \dots c_r(i_r) (|G(\theta_r)|^2)^{i_r} \right], \tag{3.44}$$

where $c_1(i_1) \dots c_r(i_r)$ are unknown constant. What we have to prove is that

1. The overall coefficient agrees with (3.43)
2. $c_j(i_j) = S(0)^{i_j - 1}$.

The first point comes out from the combinatorial coefficient that takes in account all the equivalent way to link the particles and the operator. In $\mathcal{O}_{[m,0]}$, we have an overall coefficient that is $\frac{1}{m!^2}\frac{1}{2^{2m}}$: the factorial comes from the exponentials while we get the $\frac{1}{2}$ from the integration measure that is $\frac{d\theta}{4\pi}$ and not $\frac{d\theta}{2\pi}$. We can choose the r particles in the bra (and the r in the ket) connected to $\mathcal{O}$ in

$$\left[\frac{2m\,2(m-1)\dots 2(m-r-1)}{r!}\right]^2 \tag{3.45}$$

ways. We remind that if a particle is connected to $\mathcal{O}$, its pair companion cannot be directly connect to the operator, otherwise the contribution is time dependent hence it goes to zero for long times. We have to determine in how many ways we can connect the particle in the bra with rapidity θ_1 to the others in order to have a term like $(|G(\theta_1)|^2)^{i_1}$. The answer is

$$\left[2(m-r)\,2(m-r-1)\dots 2(m-r-i_1+1)\right]^2 r\,, \tag{3.46}$$

where the r comes out from the r equivalent ways to choose a particle in the ket connected to $\mathcal{O}$. If we repeat the same argument for all the particles, we end up with an overall coefficient that is

$$\frac{1}{m!^2}\,\frac{1}{2^{2m}}\,\frac{2^{2r}}{r!^2}\,m!^2\,2^{2(m-r)}\,r! = \frac{1}{r!}\,, \tag{3.47}$$

in agreement with 3.43.

Finally, we show that $c_j(i_j) = S(0)^{i_j-1}$. Since we know that every permutation of particles gives the same contribution, it is sufficient to show it only for one of the many equivalent ways to link particles. In particular, we will consider the following way to separate the rapidities in two sets

$$\begin{aligned}
&\langle \theta_m, -\theta_m, \dots, \theta_1, -\theta_1|\mathcal{O}| - \theta'_1, \theta'_1, \dots - \theta'_m, \theta_m, \rangle = \\
&= S_{AA_1} S_{BB_1} \langle \theta_r^+, \dots \theta_1^+ |\mathcal{O}| \theta'_1, \dots \theta'_r\rangle \cdot \\
&\langle \theta_m, -\theta_m, \dots, \theta_{r+1}, -\theta_{r+1}, -\theta_r, \dots - \theta_1| - \theta'_1, \dots - \theta'_r, -\theta'_{r+1}, \theta'_{r+1}, \dots, -\theta_m,' \theta'_m\rangle\,.
\end{aligned} \tag{3.48}$$

An useful trick is to remember that the contractions are such to have, at the end, $\theta_i = \theta'_i$ for $i = 1, \dots r$. When we impose this condition, we see that the S matrices in (3.48) reduces to the identity. Now, we want to contract $-\theta_1$ with $-\theta'_{r+1}$, obtaining a $\delta(\theta_1 - \theta'_{r+1})$ and the desidered $|G(\theta_1)|^2$. Then, we link $-\theta_{r+1}$ to θ'_{r+1}. This contraction gives us a $\delta(\theta_1 + \theta_{r+1})$. Finally, we commute θ_{r+1} (that now is equal to $-\theta_1$) with $-\theta_r \ldots - \theta_2$. In this way, we end up in a situation similar to the initial one. We have two pair less, a $|G(\theta_1)|^2$ overall and a huge product of S matrices that comes from all the exchanges done. However, if we remind that at the end of our calculation we have $\theta_i = \theta'_i$ for $i = 1, \ldots r$, it is easy to see that this product of S matrices reduces to $S(0)$. We can repeat this procedure until the overall coefficient is $(S(0)|G(\theta_1)|^2)^{i_1 - 1}$. Then we contract $-\theta_1$ with $-\theta'_1$, obtaining another $|G(\theta_1)|^2$ and we go on doing the same manipulations on $-\theta_2$. It is clear that at the end we obtain exactly 3.43.

3.4 A simple example

It is instructive to see how the general ideas of the previous section apply in the simplest case provided by the one point function of the ϵ operator of the Ising model (2.84). Indeed, for this operator we can calculate exactly its one point function for any time with elementary techniques. From the form factors (2.84), it follows that the operator ϵ is a quadratic form in the creation - annhilation operators

$$\begin{aligned}\epsilon(0) = \int \frac{d\beta_1}{2\,\pi}\frac{d\beta_1}{2\,\pi}\Bigg\{ & 2\,\pi\, m \cosh\left(\frac{\beta_1 - \beta_2}{2}\right) Z^\dagger(\beta_1) Z(\beta_2) + \\ + & \left[i\,\pi\, m \sinh\left(\frac{\beta_1 - \beta_2}{2}\right)\right] \left[Z(\beta_1) Z(\beta_2) + Z^\dagger(\beta_1) Z^\dagger(\beta_2)\right] \Bigg\}. \end{aligned} \tag{3.49}$$

Since the theory is free, we can easily calculate the expectation value of binomials of the creation-annihilation operators on the boundary states (introducing, for instance, a generating functional) and we have that

$$\frac{\langle B(t)| Z^\dagger(\beta_1)\, Z(\beta_2) | B(t)\rangle}{\langle B|B\rangle} = 2\,\pi\,\delta(\beta_1 - \beta_2)\,\frac{|G(\beta_1)|^2}{1 + |G(\beta_1)|^2}, \tag{3.50}$$

$$\frac{\langle B(t)| Z(\beta_1)\, Z(\beta_2) | B(t)\rangle}{\langle B|B\rangle} = 2\,\pi\,\delta(\beta_1 + \beta_2)\,\frac{G(\beta_1) e^{-2\,i\,t\,E_1(\beta_1)}}{1 + |G(\beta_1)|^2}, \tag{3.51}$$

Hence

$$\begin{aligned}\langle \epsilon(x,t)\rangle_B = \quad & +2\,\pi\, m \int \frac{d\theta}{2\pi}\frac{|G(\theta)|^2}{1 + |G(\theta)|^2} + \\ & -2\,\pi\, m \int \frac{d\theta}{2\pi}\frac{\sinh(\theta)\, Im\left[G(\theta)\, \exp\left(-2\,i\,t E_1(\theta)\right)\right]}{1 + |G(\theta)|^2}. \end{aligned} \tag{3.52}$$

The results (3.52) has all the features we expect to hold in the general case. First of all, the time dependent part goes to zero as a consequence of the fast oscillation of the integrand. The long time asymptotic value obviously agrees with our general result: in this case, the structure of the operator is so simple that the entire sum reduces to a single term. More important, since we are able to calculate exactly the time dependence, we can also estimate the approach to the $t \to +\infty$ limit value using a stationary phase approximation. It turns out that (3.52) approaches its asymptotic value as an inverse power law, in contrast with the exponential decay of the massless case [31, 32]. Formula (3.52) is actually the continuum limit of the early result for the one point function of the transverse magnetization of the quantum Ising mode obtained in [58].

3.5 Further Works on Quantum Quenches and Integrability

Quantum quenches and integrability are quite a hot topic nowadays. In this section, we would like to highlight two recent contributions [28, 47] that are deeply connected to our work.

Divergencies and the LeClair-Mussardo formula In the previous sections we have emphasized that our treatment of the divergencies is not rigorous, even if well motivated. Indeed, when we developed these ideas the LeClair-Mussardo formula (3.21) was still a conjecture. However, some months after the pubblication of [1], this formula was proved in a remarkable paper by Balázs Pozsgay [47]. The basic idea behind this proof is the following. When we want to evaluate a thermodynamical average (e.g. a thermal one) , the brute force approach involves a summation over all the state of the system. However, in the thermodynamical limit, the only important contribution comes from the "typical" states, that correspond to the saddle point of the statistical weights. In an integrable model, these typical states are characterized by a density of roots that is given by the TBA equation. So, we could choose one of these typical states, with a large number of particle N in a large volume L, evaluate the expectation value of an operator over this typical state, and then take the thermodynamic limit. It turns out that it is actually possible to evaluate explicitly this limit and the final result is in agreement with the LeClair-Mussardo conjecture (3.21). However, we would like to stress that this result holds only for one-point functions: for multipoint correlators a complete form factors expansion, in

the spirit of the LeClair-Mussardo formula, is still lacking.
Moreover, in [47] the author considered also the quench problem from an integrable state, proving that, if we assume that the long time limit is described by the diagonal ensemble (an assumption we didn't need), then it follows our formula (3.9) with the TBA-like dressing (3.10). This is indeed a very nice check of our result, since it doesn't rely on the regularization scheme we used.

Exact Results for the Quantum Ising Chain In a very nice paper [28], Calabrese, Essler and Fagotti considered a quench of the transverse magnetic field for the quantum Ising chain. Remarkably, they were able to compute the long time limit of the one and two point function of the order parameter, showing that it is described by a generalized Gibbs ensemble. These results where obtained with two complementary approach: an evaluation of the asymptotics of Toeplitz determinant and a form factors sum, quite close to the techniques we used. So, these results emphasize that, at least for the initial states considered, the generalized Gibbs ensemble is quite a natural concept, even for semi-local operators. Moreover, since their approach is free of the technical difficulties we encountered and do not rely on any regularization procedure, it is another independent check of the correctness of our assumptions.

Chapter 4

Transformations of the Zamolodchikov-Fadeev Algebra

In this section we would like to expose some intriguing ideas about a new approach to the study of quantum quenches in integrable field theories, based on the transformations that preserve the Zamolodchikov-Fadeev algebra [2]. Admittedly, we have not yet completed our program and so we do not have conclusive results. However, we believe that it is worthwhile to explain our novel ideas and discuss some of our results.
This chapter is organized as follows. In sec. 4.1, we explains the basic ideas behind our approach and the reasons why we believe it is an interesting way to study quantum quenches. Then, in sec. 4.2, we discuss some of our results.

4.1 Introduction

One of the main open problems in the study of quantum quenches is to express the initial state $|\psi_0\rangle$ as a superposition of the eigenstates $|n\rangle$ of the post-quench Hamiltonian H, i.e. how to determine the c_n in

$$|\psi_0\rangle = \sum_n c_n |n\rangle. \tag{4.1}$$

Once such a decomposition is known, it is obviously possible to study the dynamics of the system. The brute force approach consists in computing all the inner product $\langle\psi_0|n\rangle$: in practice, however, this approach is not very fruitful and it cannot be carried out analytically. A possible way out is suggested by free systems. As we have seen in sec. 1.2.2 , a mass quench in a free bosonic or fermionic is described by a (bosonic or fermionic) Bogoliubov transformation between the pre-quench annihilation/creation operators $A_0(p), A_0^\dagger(p)$

and the post quench ones $A(p), A^\dagger(p)$. Once we know this transformation, it is possible to express the pre-quench ground state in terms of the new basis, since it must satisfy the equation $A_0(p)|\psi_0\rangle = 0$.
It is quite tempting to explore a similar strategy for interacting integrable systems. We know that the creation/annihilation operators satisfy the Zamolodchikov-Fadeev algebra, that is an extension of the commuting/anticommuting relations that hold for free systems. For example, we could consider a system described by a integrable field theory and then quench one of its parameters. Therefore, the S matrix changes from S_0 to S, and the corresponding transformation of the annihilation/creation operators should be consistent with pre-quench and post quench Zamolodchikov-Fadeev algebras.Therefore, our strategy consists in two steps:

1. Identify the possible transformations that satisfy the Zamolodchikov-Fadeev algebra. We could expect that most of this transformation are highly non linear, but it is conceivable that there exists some physically meaningful transformation that is not too complicated, at least for an infinitesimal quench.

2. Once such a transformation is identified, we could see if it is possible to use this information to write down the initial state in terms of the new quasiparticles.

We emphasize that both of these steps are non-trivial, and it is still unclear if this program can be carried out. However, we strongly believe that it is worth to explore this idea, for several reason. First of all, the brute force approach is usually not feasible, so another strategy is needed. This approach has the merit that it takes explicitly into account the fact that we are dealing with integrable field theories both before and after the quench, and the fact that the initial state is the vacuum, hence it is destroyed by the pre-quench annihilation operators. Finally, even if maybe it is not possible to complete all the steps of this program, the study of the transformations that preserve the Zamolodchikov-Fadeev algebra could have applications beyond quantum quenches.

4.2 Some solvable examples

So, we focus our attention on a integrable field theory with only one kind of particle. Therefore, the Zamolodchikov-Fadeev operators satisfy the relations

$$\begin{aligned}
&Z(\theta_1)\, Z(\theta_2) = S(\theta_1-\theta_2)\, Z(\theta_2)\, Z(\theta_1)\,, \\
&Z^\dagger(\theta_1)\, Z^\dagger(\theta_2) = S(\theta_1-\theta_2)\, Z^\dagger(\theta_2)\, Z^\dagger(\theta_1)\,, \\
&Z(\theta_1)\, Z^\dagger(\theta_2) = S(\theta_2-\theta_1)\, Z^\dagger(\theta_2)\, Z(\theta_1) + 2\,\pi\,\delta(\theta_1-\theta_2)\,.
\end{aligned} \tag{4.2}$$

We have seen in sec. 2.4 that, in the study of the analytical properties of integrable field theories, the rapidity θ is a convenient variable. Indeed, in the θ plane, the analytical structure of the S matrix is much simpler. However, when we deal with quantum quenches, the mass before and after the quench can be different. The trivial example is the mass quench in a free theory (sec. 1.2.2), but, for example, if we change the coupling constant in a relativistic field theory as the Sinh-Gordon model, the physical mass (2.88) changes even if the bare mass is constant. Therefore, it is convenient to use as a variable not the rapidity but the momentum, that is conserved under a quench of a global parameter. So, we have

$$Z(p) = Z\left(\theta = \text{arctanh}\left[\frac{p}{E(p)}\right]\right), \tag{4.3}$$

where $E(p) = \sqrt{p^2+m^2}$, and this set of operators $Z(p), Z^\dagger(p)$ satisfy the algebra

$$\begin{aligned}
&Z(p_1)\, Z(p_2) = S(p_1,p_2)\, Z(p_2)\, Z(p_1)\,, \\
&Z^\dagger(p_1)\, Z^\dagger(p_2) = S(p_1,p_2)\, Z^\dagger(p_2)\, Z^\dagger(p_1)\,, \\
&Z(p_1)\, Z^\dagger(p_2) = S(p_2,p_1)\, Z^\dagger(p_2)\, Z(p_1) + 2\,\pi\, E(p_1)\delta(p_1-p_2)\,.
\end{aligned} \tag{4.4}$$

Notice that the S matrix amplitude

$$S(p_1,p_2) = S\left(\theta = \left\{\text{arctanh}\left[\frac{p_1}{E(p_1)}\right] - \text{arctanh}\left[\frac{p_2}{E(p_2)}\right]\right\}\right), \tag{4.5}$$

is actually a function of the difference of the two rapidities, due to Lorentz invariance. Moreover, we have that

$$S(-p_1,-p_2) = S(p_2,p_1), \qquad S(p_1,p_2)S(p_2,p_1) = 1. \tag{4.6}$$

Therefore, our basic idea is the following: before the quench, the system is in the ground state of an integrable field theory described by the S matrix $S_0(p_1,p_2)$. Then, we quench one parameter, and so the unitary evolution of the system is governed by the integrable Hamiltonian with S matrix

$S(p_1, p_2)$. Our aim is to find out the relation between the old particle operators $Z_0(p), Z_0^\dagger(p)$ and the new ones $Z(p), Z^\dagger(p)$, or at least to spell out some constrain for such a transformation.

4.2.1 Bogoliubov Transformation

Since for free system the transformation is simply a Bogoliubov one, it is quite natural to explore if this is possible also for interacting integrable systems. So, we assume that

$$Z_0(p) = \alpha(p)Z(p) + \beta(p)Z^\dagger(-p), \tag{4.7}$$

where $\alpha(p), \beta(p)$ are unknown complex functions. Therefore, we have that

$$\begin{aligned} Z_0(p_1)Z_0(p_2) = \alpha(p_1)\alpha(p_2)Z(p_1)Z(p_2) + \beta(p_1)\beta(p_2)Z^\dagger(-p_1)Z^\dagger(-p_2) + \\ +\alpha(p_1)\beta(p_2)Z(p_1)Z^\dagger(-p_2) + \beta(p_1)\alpha(p_2)Z^\dagger(-p_1)Z(p_2). \end{aligned} \tag{4.8}$$

However, we immediately notice that there is something odd in this expression: we can exchange $Z_0(p_1)$ and $Z_0(p_2)$ with the pre-quench S matrix $S_0(p_1, p_2)$, while in the commutation relations of $Z(p)$ the post quench S matrix $S(p_1, p_2)$ enters. So, let us consider the state

$$|2\rangle = \int_{-\infty}^{+\infty} \frac{dq_1}{2\pi E(q_1)} \int_{-\infty}^{q_1} \frac{dq_2}{2\pi E(q_2)} \psi(q_1, q_2) Z^\dagger(q_1) Z^\dagger(q_2)|0\rangle, \tag{4.9}$$

where $|0\rangle$ is the ground state of the post-quench Hamiltonian. So, we have that $(p_1 > p_2)$

$$\begin{aligned} \langle 0|Z_0(p_1)Z_0(p_2)|2\rangle = \alpha(p_1)\alpha(p_2)\langle 0|Z(p_1)Z(p_2)|2\rangle = \\ = \alpha(p_1)\alpha(p_2)\psi(p_1, p_2)S(p_1, p_2). \end{aligned} \tag{4.10}$$

However, we know that

$$\begin{aligned} \langle 0|Z_0(p_1)Z_0(p_2)|2\rangle = S_0(p_1, p_2)\langle 0|Z_0(p_2)Z_0(p_1)|2\rangle = \\ S_0(p_1, p_2)\alpha(p_1)\alpha(p_2)\langle 0|Z(p_2)Z(p_1)|2\rangle = \alpha(p_1)\alpha(p_2)\psi(p_1, p_2)S_0(p_1, p_2), \end{aligned} \tag{4.11}$$

hence we have $S_0(p_1, p_2) = S(p_1, p_2)$. Similarly, we could consider $\langle 2|Z_0(p_1)Z_0(p_2)|0\rangle \sim \langle 2|Z(-p_1)Z(-p_2)|0\rangle$, hence we get $S_0(p_1, p_2) = S(p_2, p_1)$. Therefore, we have that

$$S_0(p_1, p_2)^2 = S(p_1, p_2)^2 = 1, \tag{4.12}$$

hence the theories before and after the quench are free ones[1]. This is the first of our results, even if it is negative: a Bogoliubov transformation can describe a quench only in free systems.

4.2.2 Infinitesimal Quantum Quenches

Instead of considering the most general case, we limit ourself to an infinitesimal transformation. Therefore, we assume to have a small parameter ϵ that is related to the variation of the parameter in the Hamiltonian. The particle operators change as

$$Z(p) = Z_0(p) + \epsilon W(p) + \dots, \tag{4.13}$$

where we have omitted terms of higher order in ϵ, while the S matrix changes as

$$S(p_1, p_2) = S_0(p_1, p_2) + \epsilon\, T(p_1, p_2) + \dots. \tag{4.14}$$

Obviously, T is a function of the difference of the two rapidities, as for the S matrix. Notice that the unitarity condition $S(p_1, p_2)S(p_2, p_1) = S(p_1, p_2)\overline{S}(p_1, p_2) = 1$ implies that

$$T(p_2, p_2) = \overline{T}(p_1, p_2) = -T(p_1, p_2)S_0^2(p_2, p_1)\,. \tag{4.15}$$

The new particle operators $Z(p), Z^{\dagger}(p)$ must satisfy the Zamolodchikov-Fadeev algebra at the first order in ϵ, and so

$$\begin{aligned} W(p_1)Z_0(p_2) + Z_0(p_1)W(p_2) &= S_0(p_1, p_2)\left[Z_0(p_2)W(p_1) + W(p_2)Z_0(p_1)\right] + \\ &+T(p_1, p_2)Z_0(p_2)Z_0(p_1) \end{aligned} \tag{4.16}$$

$$\begin{aligned} W(p_1)Z_0^{\dagger}(p_2) + Z_0(p_1)W^{\dagger}(p_2) &= S_0(p_2, p_1)\left[Z_0^{\dagger}(p_2)W(p_1) + W^{\dagger}(p_2)Z_0(p_1)\right] + \\ &+T(p_2, p_1)Z_0^{\dagger}(p_2)Z_0(p_1) + 2\,\pi\,\delta E(p_1)\,\delta(p_1 - p_2), \end{aligned} \tag{4.17}$$

where $\delta E(p1)$ is the shift of the single particle $E(p)$ energy due to the (eventual) change of the physical mass.

Now, we would like to investigate the following issue: there exists a simple transformations that satisfy the Zamolodchikov-Fadeev algebra at the infinitesimal level? In principle, it is not clear that this is the case: we could

[1]Technically, the above equations admit also the solution $\alpha(p) = 0$ or $\beta(p) = 0$. However, $\beta(p) = 0$ is not interesting, since it is simply a rescaling of the particle operators, while we do not admit $\alpha(p) = 0$ as a solution since in the limit when the variation of the parameter is zero, the transformation of the particle operators must be to the identity.

have that the only way to satisfy the algebra is to have an infinite series of operators, also at the first order in ϵ. Indeed, we have see that the conditions imposed by the Zamolodchikov-Fadeev algebra are quite strict, since in the interacting case there is no way to satisfy them with a Bogoliubov transformation.
In order to understand more carefully this point, let us consider the following ansatz

$$W(p) = \left[\int \frac{dq}{2\pi E(q)} \alpha(q,p) Z_0^\dagger(q) Z_0(q)\right] Z_0(p). \tag{4.18}$$

Clearly, this expression respect the momentum conservation. The reason behind the choice of this particular form for the transformation will be evident in a few moments: essentially, in order to reproduce the term proportional to $T(p_1,p_2)$ in (4.16) we need at least 1 $Z_0^\dagger$ and two Z_0's. We have that

$$W(p_1)Z_0(p_2) = S_0(p_1,p_2)\left[Z_0(p_2)W(p_1) + \alpha(p_2,p_2)Z_0(p_2)Z_0(p_1)\right]; \tag{4.19}$$

$$\begin{aligned} W(p_1)Z_0^\dagger(p_2) &= S_0(p_2,p_1)\left[Z_0^\dagger(p_2)W(p_1) + \alpha(p_2,p_1)Z_0^\dagger(p_2)Z_0^\dagger(p_1)\right] + \\ &+ 2\pi\, E(p_1)\delta(p_1-p_2)\left[\int \frac{dq}{2\pi E(q)}\alpha(q,p)Z_0^\dagger(q)Z_0(q)\right]. \end{aligned} \tag{4.20}$$

So, from (4.16) we have that

$$T(p_1,p_2) = S_0(p_1,p_2)\left[\alpha(p_1,p_2) - \alpha(p_2,p_1)\right], \tag{4.21}$$

while from (4.17) we get

$$\alpha(q,p) + \overline{\alpha}(q,p) = 0; \tag{4.22}$$

$$T(p_1,p_2) = S_0(p_1,p_2)\left[\alpha(p_1,p_2) + \overline{\alpha}(p_2,p_1)\right]. \tag{4.23}$$

Nicely, these conditions could be simultaneously satisfied if $\alpha(p_1,p_2)$ is purely imaginary. Notice that if $\alpha(p_1,p_2)$ is purely imaginary, $T(p_1,p_2)$ satisfies also the unitarity condition (4.15). Therefore, we have found a simple transformation that (up to the first order in ϵ) preserves the Zamolodchikov-Fadeev algebra. Notice that this transformation doesn't change the mass of the particle $\delta E(p) = 0$. Under this transformation, the ground state is invariant, since it has a Z_0 to the right that clearly annihilate the pre-quench vacuum. However, this transformation is still interesting in our opinion. Indeed, we can decompose any infinitesimal transformation that changes the ground state and the S matrix in a part that changes only the ground state and one as (4.18) that is responsible for the shift of the S matrix. This is one of the ideas that we are currently pursuing in this area. Moreover, we are complementing our abstract analysis of the transformations that preserve the Zamolodchikov-Fadeev algebra with the examination of some concrete example, e.g. infinitesimal quenches in the Sinh-Gordon model.

Conclusions

In this thesis we have explored the theme of integrability and out of equilibrium dynamics. Our attention has been focused on the simplest paradigm of out of equilibrium coherent dynamics, the so called quantum quench.

In chapter 3, motivated by novel experimental results [6], we have addressed the issue of thermalization in integrable systems. On the one hand, we have shown what is probably the first argument in favor of a generalized Gibbs ensemble for interacting integrable one-dimensional systems. On the other hand, our work is quite interesting also from the technical point of view: for the first time the calculation of the long time limit of the expectation value of a local operator has been performed in the context of integrable field theories. The validity of some of our assumptions, that were well justified but not rigorously proven, has been confirmed by subsequent works [28, 47].

Instead, in chapter 4, we have studied a complimentary problem: rather than studying the dynamics, assuming a specific initial state, we have tried to take advantage of the integrability before and after the quench in order to write down an explicit formula for the initial state. Admittedly, our results are not conclusive and there is room for many further developments. However, we believe that the key concept of the transformations that preserve the Zamolodchikov-Fadeev algebra could play a major role in the next years, also beyond the study of quantum quenches.

In the classical book "Arnold's Problems" [59] Vladimir Arnol'd wrote that:

"Poincaré used to say that precise formulation, as a question admitting a "yes or no" answer, is possible only for problems of little interest. Questions that are really interesting would not be settled this way: they yield gradual

forward motion and permanent development."

In our opinion, this is the status of the problem of thermalization in quantum systems. The intense work of the community in the last few years has indeed provided nice new results and intuitions, but many aspects of this problem still need a thoughtful investigation. Among the others, we would like to emphasize the problem of small perturbation of integrable systems: in our minds we have this idea, that there is a qualitatively different behavior between integrable and not integrable systems, and actually the results of the last few years seem to confirm this intuition. However, what happens when we add a small non integrable perturbation to our system? Is the integrable structure immediately destroyed or not? This point is of course very important for experiments, since there is always a deviation from integrability, for example, a trapping potential. Moreover, this is the quantum analog of the Fermi-Pasta-Ulam problem [60, 61], that has stimulated a lot of research works in classical mechanics as well as in statistical physics over the last fifty years. Quite ironically, some of the early investigators of the Fermi-Pasta-Ulam problem thought that it could provide an alternative to the quantum mechanical description of macroscopic bodies at low temperature, in the spirit of an earlier proposal by Jeans (see e.g. [62,63]).

For these reasons, we believe that in the next few years a proper understanding of the transition between integrable and non integrable systems will provide to be a major focus of the research in non equilibrium coherent dynamics. Moreover, we hope that our research work in the area of integrable field theories will provide to be a suitable starting point for such an investigation, for example relying on the perturbative theory already developed [64, 65].

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