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Matrix elements from algebraic Bethe anzatz: novel applications in statistical physics

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Introduction

Happy families are all alike
(Lev Tolstoy, Anna Karenina)

Mathematical techniques in statistical physics have developed in recent years along several directions. In particular, *integrable* models have received considerable attention because they represent prototypical examples which can be exactly solved, either to gain some intuition about the behaviour of more complex systems or to approximate a class of real systems in certain limits.

Models of interest in this thesis are constituted by a large number of quantum degrees of freedom, which either will be or will be reducible in the form of *spins*. Moreover, a key role in our analyses will be the fact that they are based on *exactly solvable* models. The first exact solution of a quantum statistical system goes back to Bethe's treatment of the one-dimensional Heisenberg spin chain [1]. He found that the eigenvectors of this hamiltonian are written is a certain specific form (Bethe's ansatz), and that the eigenvalues are described by a system of algebraic equations nowadays known as Bethe's equations.

Bethe's technique has been systematized and applied to various other models, including two-dimensional models of classical statistical mechanics, allowing the exact computation of the low-lying eigenvalues and of the free energy and the access to thermodynamic properties, such as phases [2].

Through all these developments, it was recognized that at the heart of the method there is a single algebraic relation among the "building blocks" of the models, called *Yang-Baxter equation*. Schematically, each solution of this equation gives rise to a class of exactly solvable hamiltonians. The way in which these hamiltonians are written and diagonalized has been systematized in the so-called quantum inverse scattering method [3] or algebraic Bethe ansatz [4].

For two-dimensional systems at the critical point that identifies in the phase space a second order phase transition, a totally different approach, based on the effective description in terms of a *conformal field theory*, was initiated in [5]. In this situation, an infinite-dimensional algebra constrains the theory and allows its exact solution, which amounts to the computation of multi-point correlation functions. It was soon realized that away from the critical point, specific field theories admit a description in terms of particles, whose interaction can be summarized in terms of *scattering matrices*, which are forced to satisfy the Yang-Baxter equation [6]. Also in this context, this stringent requirement allows ultimately for the exact solution of the theory.

The aim of this thesis is to provide further examples of the great impact that the techniques related to the algebraic Bethe ansatz to integrable models have on modern problems

of condensed matter physics, as well as to highlight the single algebraic structure that unites many of them. In facts, the relative flexibility of the underlying formalism allows somewhat "unorthodox" applications: among those illustrated in this thesis, there appear disordered and even nonintegrable hamiltonians, as well as quantum field theories on some nontrivial support. One essential characteristic of this work is the emphasis posed on *matrix elements* of quantum observables, which are explicitly obtainable by algebraic Bethe ansatz.

The first chapter is an introduction to the general concepts about integrable lattice models and to the Yang-Baxter equation. One important solution to this equation is presented, and it is shown how to derive from it two important hamiltonians, namely the Heisenberg and the Richardson hamiltonians, which will be considered extensively throughout the thesis. Moreover, the algebraic Bethe ansatz method for writing eigenstates and eigenvalues of these hamiltonians is presented in general terms.

The second chapter deals with the Richardson model, as a few-body description of Bardeen-Cooper-Schrieffer (BCS) superconductivity. In particular, the BCS equations are reviewed and shown how to be derived from the scaling limit of the model. An original investigation of the Josephson current flowing among two coupled few-body superconductors is presented, having in mind experiments featuring trapped cold-atoms with tunable interaction. The crossover among the superconducting regime and the one to a Bose-Einstein condensate (BEC) is studied. Part of the results are contained in my paper [7].

The third chapter analyses the effect of disorder on the properties of the exact many-body eigenstates of the Richardson model, with attention to their localization properties in the Hilbert space. Motivation for the work, which lies in the study of thermalization and of hopping conduction in quantum systems is presented, together with the original results of my article [8].

The fourth chapter introduces the basics concept of integrable field theory, with attention to the computation of correlation functions. The sine-Gordon/massive Thirring theories are presented and shown to be particularly relevant in condensed matter and statistical physics. This is a review chapter.

The fifth chapter tackles the problem of computing correlation functions of the sine-Gordon field theory in finite size. The technique used is based on an already known lattice regularization, which is reviewed in the first part. Then a new formula, contained in my article [9], for the generating function of connected correlation functions in finite volume is derived, using the algebraic formalism introduced earlier in this thesis.

Contents

Chapter 1

The models and the algebraic formalism

This chapter provides an introduction to the formalism of algebraic Bethe ansatz and of the models which will be used in this work. Connections among the various Hamiltonians are highlighted from the point of view of their algebraic construction.

1.1 Integrability in quantum systems

1.1.1 Spin systems

Consider a system with $2N$ sites and a spin$-\frac{1}{2}$ degree of freedom on each site. The Hilbert space of the system is spanned by the direct product of the single-site basis

$$\{|\uparrow\rangle, |\downarrow\rangle\} \tag{1.1}$$

For the time being, the geometry is that of a one-dimensional chain with periodic boundary conditions. Each site is labelled by an integer $m, n, \ldots$ and carries a local Hilbert space upon which a spin-1/2 representation of $su(2)$ acts. It is possible to carry on part of the construction without specifying a particular model. In other words, as we will see below, it is possible to diagonalize an Hamiltonian without even knowing its expression.

An essential step for constructing a lattice model which is integrable through algebraic Bethe ansatz amounts to finding a matrix that satisfies the Yang-Baxter equation (YBE). Given three "local" Hilbert spaces a, b, c, the so–called R-matrix satisfies:

$$R_{ab}(\lambda)R_{ac}(\mu)R_{bc}(\lambda-\mu) = R_{bc}(\lambda-\mu)R_{ac}(\mu)R_{ab}(\lambda) \tag{1.2}$$

for any choice of the complex "rapidities" λ and μ. Here and in the following, whenever useful, matrix representations will carry subscripts with local space labels. The Yang-Baxter relation is fundamental in quantum integrable systems, and may be taken as a definition (see also [10]) of quantum integrability itself. The analogue formulation in quantum field theory will be reviewed section 4.2. It consists on a relation on the two-paricle S–matrix of the theory and can be seen as a necessary condition on the factorization of the full S–matrix into two–body processes.

There is an analogous formulation of the relation above for open boundary conditions, which takes the name of boundary Yang-Baxter equation [11] and holds in the same form in the context of integrable quantum field theories [12].

Another property (see, e.g., [13],[14]) of the R-matrix is (provided $b(\lambda) \neq \pm c(\lambda)$), *unitarity*:

$$R_{ab}(\lambda - \mu) R_{ba}(\mu - \lambda) = \mathbf{1} \ , \tag{1.3}$$

which follows from inverting one matrix in (1.2) and applying the permutation operator. This requirement is functional to the construction and diagonalization of integrable Hamiltonians through algebraic Bethe ansatz and will find application later on in section 1.3.1.

Some classes of solutions to (1.2) are know [1] . The simplest and by far the best known is the R-matrix associated with the six-vertex model and with the corresponding quantum chain [2], upon which the models analysed in this thesis are based.

We write here the "six-vertex" and the "rational" R-matrix, represented on the local $\mathcal{C}^2$ spaces α and β:

$$R(\lambda)_{\alpha,\beta} = \begin{pmatrix} 1 & & & \\ & b(\lambda) & c(\lambda) & \\ & c(\lambda) & b(\lambda) & \\ & & & 1 \end{pmatrix}_{\alpha,\beta} \tag{1.4}$$

where the functions b and c, for the XXZ family are taken[2] as:

$$b(\lambda) = \frac{\sinh(\lambda)}{\sinh(\lambda - i\gamma)} \ , \quad c(\lambda) = \frac{\sinh(-i\gamma)}{\sinh(\lambda - i\gamma)} \tag{1.5}$$

and the *anisotropy parameter* γ is related to the coupling of the spins along the z axis.

With this choice, the matrix enjoys the property of *crossing symmetry*

$$(C \otimes \mathbf{1}) \, R_{a,b}(i\pi - \lambda) \, (C \otimes \mathbf{1}) = R^{t_1}_{a,b}(\lambda) \tag{1.6}$$

in which C is, in general, an appropriate 2×2 matrix such that $C^2 = \mathbf{1}$ and $C^t = \pm C$ and in this case $C = \sigma^x$. The superscript t_1 indicates transposition in the first of the two spaces only, in other words, the exchange of the order of $|\uparrow\rangle$ and $|\downarrow\rangle$ in the first basis.

For the *rational model*, which is the starting point for obtaining the Richardson model studied in the following, we have instead

$$b(\lambda) = \frac{\lambda}{\lambda + \eta} \ , \quad c(\lambda) = \frac{\eta}{\lambda + \eta} \tag{1.7}$$

and, similarly, the isotropic XXX is obtained by specializing the value $\eta = i$, so that we have

$$b(\lambda) = \frac{i}{\lambda + i} \ , \quad c(\lambda) = \frac{\lambda}{\lambda + i} \tag{1.8}$$

The models are not independent. The hyperbolic R-matrix (1.4,1.5) can be analytically continued to imaginary values of the anisotropy parameter

$$e^{i\gamma} \to e^{-\epsilon\eta} \tag{1.9}$$

for some small real ϵ. Then, vanishing values of the argument $e^{\lambda} \simeq 1 + \epsilon v$ and of the anisotropy parameter reduce the XXZ R-matrix to the rational form (1.7), i.e.:

$$R(u) = \frac{1}{u + \eta} \left(\eta \, \mathbf{1} + u \mathbf{P} \right) \tag{1.10}$$

[1] due to the fact that the above relation is a representation on a, b, c of a defining property the "universal R-matrix" of a quantum group, they are classified accordingly

[2] an overall multiplying function can be considered as well

which is itself a solution of the Yang-Baxter equation.

As it will be seen in Chapter 4, integrable two-dimensional quantum field theories can be formulated in terms of a two-particle scattering matrix. Integrability of the theory will imply the constraint (1.2) that such an object has to satisfy. The solution (1.4,1.5) will then be the exact two-particle scattering matrix between the fundamental excitations in the sine-Gordon field theory.

1.2 Introduction to Algebraic Bethe Ansatz

Algebraic Bethe ansatz (ABA) [15] is a powerful technique that allows to write exact eigenstates and eigenvalues of a class of lattice models. Both the effectiveness and the main limitation of this technique are not confined to the exact diagonalization of a particular Hamiltonian, but also consists in the possibility of generating a class of Hamiltonians starting from the underlying requirement of integrability itself.

There are two main classes of quantum systems that can be dealt with through the ABA, namely periodic chains and fully connected models. In the following, we shall review the construction for the most noticeable representatives for each class: the Heisenberg chain and the Richardson model. Open chains, as well as "ladders", can also be dealt with under suitable modifications.

A great advantage of the algebraic version of Bethe ansatz stays in the fact that, despite the technical difficulties, it has proven to be possible to solve the "inverse scattering" problem, at least for some simple prototypical models. This means reconstructing matrix element of operators in terms of generalized creation and destruction operators, which satisfy algebraic relations that ultimately allow the reconstruction of matrix elements of the operators. More details will be provided below.

1.2.1 The Yang-Baxter relation at work

The ingredient in the algebraic Bethe ansatz construction which actually specifies the model under study is the so-called Lax L-operator, which we will represent as a matrix valued function of a complex argument ("rapidity"). It acts on a couple of local Hilbert spaces $C^2 \otimes C^2$.

For definiteness we anticipate that for the XXZ chain we have the following form:

$$L^{XXZ}_{m,n}(\lambda) = \begin{pmatrix} q^{S^z_n - i\lambda} - q^{-S^z_n + i\lambda} & \left(q - q^{-1}\right) S^-_n \\ \left(q - q^{-1}\right) S^+_n & q^{-S^z_n - i\lambda} - q^{S^z_n + i\lambda} \end{pmatrix}_m = R_{m,n}\left(\lambda + i\frac{\gamma}{2}\right) \tag{1.11}$$

with $q = e^{-i\gamma}$, whereas for Richardson

$$L^{R}_{m,n}(\lambda) = \frac{1}{\lambda}\begin{pmatrix} \lambda + \eta S^z_n & \eta S^-_n \\ \eta S^+_n & \lambda - \eta S^z_n \end{pmatrix}_m \tag{1.12}$$

and once again the isotropic Heisenberg chain is obtained from the value $\eta = i$

$$L^{XXX}_{m,n}(\lambda) = \begin{pmatrix} \lambda + i S^z_n & S^-_n \\ S^+_n & \lambda - i S^z_n \end{pmatrix}_m \tag{1.13}$$

This operator is a function of a complex variable λ and must satisfy a specialized version of the YBE:

$$L_{ab}(\mu) L_{ac}(\lambda) R_{bc}(\lambda - \mu) = R_{bc}(\lambda - \mu) L_{ac}(\lambda) L_{ab}(\mu) \tag{1.14}$$

This relation is generally derived by (1.2) by defining L as the R-matrix itself with some shift of its argument along the imaginary axis, as it will seen below. Moreover, it is required that it reduces to a permutation between local spaces

$$P_{a,b} \left|\psi\right\rangle_a \otimes \left|\psi'\right\rangle_b = \left|\psi'\right\rangle_a \otimes \left|\psi\right\rangle_b \tag{1.15}$$

for some special value of its argument. The reason for this is in the great simplification that is obtained in the construction of the Hamiltonian (see below).

We will now define the basic operators which are necessary for constructing the Hamiltonian and the conserved charges, the latter being operators on the "physical" Hilbert space built as the closure of the tensor product of the single-site Hilbert spaces $\mathcal{C}^2 \bar{\otimes} \ldots \bar{\otimes} \mathcal{C}^2$. Introduce an auxiliary space a and consider a family of Lax operators that act on this subspace and on one (say, the n-th) of the local Hilbert spaces $L_{a,n}$, while behave as the identity on all other local spaces. Then we can define a *monodromy matrix* as follows:

$$T(\lambda) = L_{a,2N}(\lambda) L_{a,2N-1}(\lambda) \ldots L_{a,1}(\lambda) \tag{1.16}$$

We will actually need a generalization of this basic definition, obtained by introducing site-dependent shifts h_α, $\alpha = 1, \ldots, 2N$ in the arguments of our operators, called *inhomogeneities*:

$$T_a(\lambda) = K_a L_{a,2N}(\lambda - h_{2N}) L_{a,2N-1}(\lambda - h_{2N-1}) \ldots L_{a,1}(\lambda - h_1) \tag{1.17}$$

The Sklyanin's matrix K_a is present whenever the boundary conditions are not strictly periodic: instead, the local state in the m-th and the $2N + m$-th sites are identified only up to the action of this operator

$$\left|\psi_m\right\rangle = K_a \left|\psi_m\right\rangle \tag{1.18}$$

Since it is nontrivial only on the auxiliary space, such an action reduces, in the physical space, to the multiplication by a phase factor called "twist". This can also be written as

$$K_a B_m K_a^\dagger = B_{m+2N} \tag{1.19}$$

with B_m a generic operator acting on the m-th physical space. The lattice structure is still periodic, because the twist affects only the quantum mechanical properties,

Such a matrix acts nontrivially on the auxiliary space only and satisfies

$$[R_{ab}(\lambda - \mu), K_a(\lambda) K_b(\mu)] = 0 \tag{1.20}$$

For our purposes, it will be sufficient to consider a K_a as independent from the rapidity and diagonal. With this choice, the above requirement is automatically satisfied. It is convenient to specify since now the parametrization of the twist as:

$$K_a = e^{i\omega(\sigma_a^z - 1)} \tag{1.21}$$

in which the usual Pauli matrix appears.

An important property of the monodromy matrix is that it braids like (1.14), by virtue of the so-called "train argument". In facts, denoting a second copy of the auxiliary space as b, we can apply one after another the various copies of (1.14) and (1.20) as follows:

$$\begin{aligned}
&R_{ab}(\lambda - \mu) K_a L_{a,2N}(\lambda - h_{2N}) \ldots L_{a,1}(\lambda - h_1) K_b L_{b,2N}(\lambda - h_{2N}) \ldots L_{b,1}(\lambda - h_1) \\
&= R_{ab}(\lambda - \mu) K_a K_b L_{a,2N}(\lambda - h_{2N}) L_{b,2N}(\lambda - h_{2N}) \ldots L_{a,1}(\lambda - h_1) L_{b,1}(\lambda - h_1) \\
&= K_a K_b R_{ab}(\lambda - \mu) L_{a,2N}(\lambda - h_{2N}) L_{b,2N}(\lambda - h_{2N}) \ldots L_{a,1}(\lambda - h_1) L_{b,1}(\lambda - h_1) \\
&= K_b K_a L_{b,2N}(\lambda - h_{2N}) L_{a,2N}(\lambda - h_{2N}) R_{ab}(\lambda - \mu) \ldots L_{a,1}(\lambda - h_1) L_{b,1}(\lambda - h_1) \\
&\ldots \\
&= K_b K_a L_{b,2N}(\lambda - h_{2N}) L_{a,2N}(\lambda - h_{2N}) \ldots L_{b,1}(\lambda - h_1) L_{a,1}(\lambda - h_1) R_{ab}(\lambda - \mu)
\end{aligned} \tag{1.22}$$

Then we have a crucial identity in the ABA construction, named "RTT-relation" for obvious reasons

$$R_{ab}(\lambda-\mu)T_a(\lambda)T_b(\mu) = T_b(\mu)T_a(\lambda)R_{ab}(\lambda-\mu) \tag{1.23}$$

Let's now introduce a standard parametrization for the monodromy matrix:

$$T_a(\lambda) = \begin{pmatrix} \mathcal{A}(\lambda) & \mathcal{B}(\lambda) \\ \mathcal{D}(\lambda) & \mathcal{D}(\lambda) \end{pmatrix}_a \tag{1.24}$$

when is represented on the auxiliary space. The elements that appear in the matrix above are operators on the Hilbert space of the chain and will be used.

The braiding relation (1.23) fixes the relations between the elements of the monodromy matrix in terms of those of R. We write in extended form on the auxiliary spaces, for the sake of clarity, the product

$$\begin{pmatrix} 1 & & & \\ & b & c & \\ & c & b & \\ & & & 1 \end{pmatrix} \begin{pmatrix} \mathcal{A}(\lambda)\,\mathcal{A}(\mu) & \mathcal{A}(\lambda)\,\mathcal{B}(\mu) & \mathcal{B}(\lambda)\,\mathcal{A}(\mu) & \mathcal{B}(\lambda)\,\mathcal{B}(\mu) \\ \mathcal{A}(\lambda)\,\mathcal{C}(\mu) & \mathcal{A}(\lambda)\,\mathcal{D}(\mu) & \mathcal{B}(\lambda)\,\mathcal{C}(\mu) & \mathcal{B}(\lambda)\,\mathcal{D}(\mu) \\ \mathcal{C}(\lambda)\,\mathcal{A}(\mu) & \mathcal{C}(\lambda)\,\mathcal{B}(\mu) & \mathcal{D}(\lambda)\,\mathcal{A}(\mu) & \mathcal{D}(\lambda)\,\mathcal{B}(\mu) \\ \mathcal{C}(\lambda)\,\mathcal{C}(\mu) & \mathcal{C}(\lambda)\,\mathcal{D}(\mu) & \mathcal{D}(\lambda)\,\mathcal{C}(\mu) & \mathcal{D}(\lambda)\,\mathcal{D}(\mu) \end{pmatrix}$$
$$= \begin{pmatrix} \mathcal{A}(\mu)\,\mathcal{A}(\lambda) & \mathcal{B}(\mu)\,\mathcal{A}(\lambda) & \mathcal{A}(\mu)\,\mathcal{B}(\lambda) & \mathcal{B}(\mu)\,\mathcal{B}(\lambda) \\ \mathcal{C}(\mu)\,\mathcal{A}(\lambda) & \mathcal{D}(\mu)\,\mathcal{A}(\lambda) & \mathcal{C}(\mu)\,\mathcal{B}(\lambda) & \mathcal{D}(\mu)\,\mathcal{B}(\lambda) \\ \mathcal{A}(\mu)\,\mathcal{C}(\lambda) & \mathcal{B}(\mu)\,\mathcal{C}(\lambda) & \mathcal{A}(\mu)\,\mathcal{D}(\lambda) & \mathcal{B}(\mu)\,\mathcal{D}(\lambda) \\ \mathcal{C}(\mu)\,\mathcal{C}(\lambda) & \mathcal{D}(\mu)\,\mathcal{C}(\lambda) & \mathcal{C}(\mu)\,\mathcal{D}(\lambda) & \mathcal{D}(\mu)\,\mathcal{D}(\lambda) \end{pmatrix} \begin{pmatrix} 1 & & & \\ & b & c & \\ & c & b & \\ & & & 1 \end{pmatrix} \tag{1.25}$$

By equating term by term the elements of the RTT-relation, we obtain a set of quadratic algebraic relations among operators acting on the physical Hilbert space of the chain, which is named Yang-Baxter algebra:

$$[\mathcal{B}(\lambda),\mathcal{B}(\mu)] = [\mathcal{C}(\lambda),\mathcal{C}(\mu)] = 0 \tag{1.26}$$

$$[\mathcal{A}(\lambda),\mathcal{A}(\mu)] = [\mathcal{D}(\lambda),\mathcal{D}(\mu)] = 0 \tag{1.27}$$

$$\mathcal{A}(\mu)\,\mathcal{B}(\lambda) = \frac{1}{b(\lambda-\mu)}\,\mathcal{B}(\lambda)\,\mathcal{A}(\mu) - \frac{c(\lambda-\mu)}{b(\lambda-\mu)}\,\mathcal{B}(\mu)\,\mathcal{A}(\lambda) \tag{1.28}$$

$$\mathcal{D}(\lambda)\,\mathcal{B}(\mu) = \frac{1}{b(\lambda-\mu)}\,\mathcal{B}(\mu)\,\mathcal{D}(\lambda) - \frac{c(\lambda-\mu)}{b(\lambda-\mu)}\,\mathcal{B}(\lambda)\,\mathcal{D}(\mu) \tag{1.29}$$

$$[\mathcal{A}(\mu),\mathcal{D}(\lambda)] = \frac{c(\lambda-\mu)}{b(\lambda-\mu)}\,(\mathcal{B}(\lambda)\,\mathcal{C}(\mu) - \mathcal{B}(\mu)\,\mathcal{C}(\lambda)) \tag{1.30}$$

$$[\mathcal{B}(\mu),\mathcal{C}(\lambda)] = \frac{c(\lambda-\mu)}{b(\lambda-\mu)}\,(\mathcal{A}(\lambda)\,\mathcal{D}(\mu) - \mathcal{A}(\mu)\,\mathcal{D}(\lambda)) \tag{1.31}$$

$$\mathcal{C}(\lambda)\,\mathcal{A}(\mu) = \frac{1}{b(\lambda-\mu)}\,\mathcal{A}(\mu)\,\mathcal{C}(\lambda) - \frac{c(\lambda-\mu)}{b(\lambda-\mu)}\,\mathcal{A}(\lambda)\,\mathcal{C}(\mu) \tag{1.32}$$

$$\mathcal{C}(\mu)\,\mathcal{D}(\lambda) = \frac{1}{b(\lambda-\mu)}\,\mathcal{D}(\lambda)\,\mathcal{C}(\mu) - \frac{c(\lambda-\mu)}{b(\lambda-\mu)}\,\mathcal{D}(\mu)\,\mathcal{C}(\lambda) \tag{1.33}$$

The trace over the auxiliary space of a monodromy matrix defines the *transfer matrix*

$$\hat{\tau}(\lambda) = \mathrm{Tr}_a\,[T(\lambda)_a] = \mathcal{A}(\lambda) + e^{-2i\omega}\mathcal{D}(\lambda) \tag{1.34}$$

which again acts on the Hilbert space of the chain.

This object is crucial because we can write from (1.34) a set of $2N$ algebraically independent operators Q_j (as many as the size of the system) and to show that they commute. As a consequence, these objects will act diagonally on all the eigenstates of the transfer matrix. Most importantly, as long as one of these operators can be chosen as a physically meaningful Hamiltonian, our quantum system is endowed with a set of charges that commute among them and with the Hamiltonian, i.e., that are conserved in time. All such charges can be generated by differentiation

$$Q_n = \frac{\partial^n}{\partial \lambda^n} \log \tau(\lambda) \Big|_{\lambda = i\,\gamma\,/2} \tag{1.35}$$

An important property of the transfer matrix is that

$$[\hat{\tau}(\lambda), \hat{\tau}(\mu)] = 0\,, \qquad \forall \lambda\,,\ \mu \tag{1.36}$$

which can be proven starting from (1.23) and inverting the R-matrix

$$R_{ab}(\lambda - \mu) T_a(\lambda) T_b(\mu) R_{ab}(\lambda - \mu)^{-1} = T_b(\mu) T_a(\lambda) \tag{1.37}$$

and then tracing over the space $a \otimes b$ and using the cyclic property of the trace

$$\mathrm{Tr}_{a\otimes b} \left[R_{ab}(\lambda - \mu) T_a(\lambda) T_b(\mu) R_{ab}(\lambda - \mu)^{-1} - T_b(\mu) T_a(\lambda) \right] = \left[\mathrm{Tr}_a \left[T_a(\lambda) \right], \mathrm{Tr}_b \left[T_b(\mu) \right] \right] \tag{1.38}$$

which proves (1.36). From this property, which must be true for arbitrary choices of the spectral parameters λ, μ, it follows that all the charges defined in (1.35) commute among them.

1.2.2 Construction of the eigenstates

Our goal is to construct a set of eigenstates of the transfer matrix since these states are simultaneous eigenstates of the Hamiltonian and of all the conserved charges.

Within the formalism of second quantization, a state with, say, n free fermions on momentum levels $\mathbf{k}_1, \ldots, \mathbf{k}_n$, is created by the action of ladder operators $c^\dagger_{\mathbf{k}_1}, \ldots, c^\dagger_{\mathbf{k}_n}$ on a vacuum state $|0\rangle$. Such a state is defined to be the one which is annihilated by all the lowering operators, i.e., by $c_{\mathbf{k}_j} |0\rangle = 0\ \forall j$. The framework of ABA somehow extends these notions to strongly interacting systems. The key idea is that an eigenstate can be built as the repeated action of generalized ladder operators, the role of which will be played by the $\mathcal{B}$s in (1.24), on a reference state:

$$|\Psi(\{\lambda\})\rangle = \mathcal{B}(\lambda_1) \ldots \mathcal{B}(\lambda_n) |ref\rangle \tag{1.39}$$

where the interacting nature of the system is encoded in the fact that the arguments of the ladder operator are not independent one from another, but must satisfy a system of n algebraic equations, which will be written below.

As for the reference state, also called "pseudovacuum", it is defined by the property:

$$\mathcal{C}(\lambda) |ref\rangle = 0 \tag{1.40}$$

In order to construct such a vector, we require that the action of the monodromy matrix (1.24) to be:

$$T(\lambda) |ref\rangle = \begin{pmatrix} a(\lambda) & whatever \\ 0 & d(\lambda) \end{pmatrix} |ref\rangle \tag{1.41}$$

From the specific spin models and L-matrices that are under study, it is clear that if we chose every local variable to be in the "spin-up" state $|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ then the Lax operator will act as a triangular matrix on it

$$L_{a,m}(\lambda - h_m)\,|\uparrow\rangle = \begin{pmatrix} \alpha(\lambda - h_m)\,|\uparrow\rangle & \neq 0 \\ 0 & \delta(\lambda - h_m)\,|\uparrow\rangle \end{pmatrix}_a \tag{1.42}$$

Multiplication of many of these upper-triangular matrices in (1.17) on the different sites will preserve the upper-triangularity, leading to (1.41).

The action of the transfer matrix on this state is then simply written as

$$\hat{\tau}(\lambda)\,|ref\rangle = \left(\prod_{l=1}^{2N}\alpha(\lambda - h_l) + e^{-2i\omega}\prod_{l=1}^{2N}\delta(\lambda - h_l)\right)|ref\rangle \equiv (a(\lambda) + d(\lambda))\,|ref\rangle \tag{1.43}$$

and defines the functions[3] a and d that are to be used in the following. For the sake of definiteness, we write them here for the XXZ model:

$$a(\lambda) = \prod_{m=1}^{2N}\frac{\sinh\left(\lambda - h_m + i\frac{\gamma}{2}\right)}{\sinh\left(\lambda - h_m\right)}\,, \qquad d(\lambda) = \prod_{m=1}^{2N}\frac{\sinh\left(\lambda - h_m - i\frac{\gamma}{2}\right)}{\sinh\left(\lambda - h_m\right)} \tag{1.44}$$

and for the rational case:

$$a(\lambda) = \prod_{m=1}^{2N}\frac{\lambda - h_m - \frac{\eta}{2}}{\lambda - h_m}\,, \qquad d(\lambda) = \prod_{m=1}^{2N}\frac{\lambda - h_m + \frac{\eta}{2}}{\lambda - h_m}\,, \tag{1.45}$$

and the XXX chain, that can be obtained from this expression for $\eta = -i$:

$$a(\lambda) = \prod_{m=1}^{2N}\frac{\lambda - h_m + \frac{i}{2}}{\lambda - h_m}\,, \qquad d(\lambda) = \prod_{m=1}^{2N}\frac{\lambda - h_m - \frac{i}{2}}{\lambda - h_m}\,, \tag{1.46}$$

We turn on the action of the two parts of the transfer matrix on a generic Bethe state. Using the exchange relations (1.26), we see that:

$$\begin{aligned}\mathcal{A}(\mu)\,\mathcal{B}(\lambda_1)\ldots\mathcal{B}(\lambda_M)\,|ref\rangle = &\prod_{l=1}^{M} b^{-1}(\lambda_l - \mu)a(\mu)\,\mathcal{B}(\lambda_1)\ldots\mathcal{B}(\lambda_M)\,|ref\rangle + \\ &+\sum_{l=1}^{M} C_l(\mu|\{\lambda\})\,\mathcal{B}(\lambda_1)\ldots\widehat{\mathcal{B}(\lambda_l)}\ldots\mathcal{B}(\lambda_M)\,\mathcal{B}(\mu)\,|ref\rangle\end{aligned} \tag{1.47}$$

where the symbol $\widehat{\ }$ denotes ellipsis. The first term gives back the original state, multiplied by a factor, while the other terms have the same number of rapidities, but one of those relative to the state has been replaced by the argument of the operator. Because of the commutativity of the $\mathcal{B}$s among them, we can focus on one term only and obtain all the others by permutation. If in the exchange with $\mathcal{B}(\lambda_1)$ one takes the second term in (1.26), then a factor

$$C_1(\mu|\{\lambda\}) = -\frac{c(\lambda_1 - \mu)}{b(\lambda_1 - \mu)}\prod_{j=2}^{M} b^{-1}(\lambda_l - \lambda_1)a(\lambda_1)$$

[3] actually, the *ratio* of the two is determined

is generated, multiplying the state in which the rapidity μ replaces λ_1. This state is not generated by any other exchange, hence commutativity of the $\mathcal{B}$s imposes that

$$C_l(\mu|\{\lambda\}) = -\frac{c(\lambda_l - \mu)}{b(\lambda_l - \mu)} \prod_{j\neq l}^{M} b^{-1}(\lambda_j - \lambda_l) a(\lambda_l) \tag{1.48}$$

We can focus on the action of the other part of the operator and compute similarly:

$$\mathcal{D}(\mu)\,\mathcal{B}(\lambda_1)\dots\mathcal{B}(\lambda_M)\,|ref\rangle = \prod_{l=1}^{M} b^{-1}(\mu - \lambda_l) d(\mu)\,\mathcal{B}(\lambda_1)\dots\mathcal{B}(\lambda_M)\,|ref\rangle + \\ + \sum_{l=1}^{M} \tilde{C}_l(\mu|\{\lambda\})\,\mathcal{B}(\lambda_1)\dots\widehat{\mathcal{B}(\lambda_l)}\dots\mathcal{B}(\lambda_M)\,\mathcal{B}(\mu)\,|ref\rangle \tag{1.49}$$

and once again we have M unwanted terms, one for each root of the state, multiplied by a factor

$$\tilde{C}_l(\mu|\{\lambda\}) = -\frac{c(\mu - \lambda_l)}{b(\mu - \lambda_l)} \prod_{j\neq l}^{M} b^{-1}(\lambda_l - \lambda_j) d(\lambda_l) \tag{1.50}$$

In order to obtain a diagonal action of the element of the transfer matrix, we add (1.47) to (1.49) and ask that all unwanted terms cancel. This produces a system of M algebraic equations for the M unknown rapidities $\{\lambda\}$ that form the state:

$$a(\lambda_l) \prod_{j\neq l}^{M} b^{-1}(\lambda_j - \lambda_l) - e^{-2i\omega} d(\lambda_l) \prod_{j\neq l}^{M} b^{-1}(\lambda_l - \lambda_j) = 0 \qquad j = 1, \dots, M \tag{1.51}$$

These are the Bethe equations (BE) for a state defined by a set of M rapidities. Of course, various solutions with the same number of roots can exist.

To associate a physical meaning to this number, we need to remind that we are constructing spin models, and specifically from the Lax operators (1.11,1.12,1.13). It is easily seen that the reference state will be the state in which all the local spin variables have a definite orientation, pointing upwards along the z axis

$$|ref\rangle = |\uparrow, \dots, \uparrow\rangle \tag{1.52}$$

Then, one can see from the form of the Bethe ansatz ladder operators (see below) that the total magnetization of the state is:

$$S = N - \frac{M}{2} \tag{1.53}$$

To rephrase, to find all the states with given magnetization S, one should find all the solutions of (1.51) with $M = 2(S - M)$ rapidities. Completeness of the set of all solutions of the Bethe equations is shown by direct inspection for small size systems, but there is no available proof for generic sizes.

1.3 The Heisenberg chain

The Heisenberg chain is a paradigmatic model for ferromagnetism in one dimension. The Hamiltonian reads:

$$H_{xxz} = -J \sum_{n=1}^{2N} [S_n^x S_n^x + S_n^y S_n^y + \cos\gamma S_n^z S_n^z] \tag{1.54}$$

with the anisotropy parameter γ introduced earlier.

There are several physical systems that can be modeled by this kind of Hamiltonian. For example, the structure of $KCuF_3$ is such that the unpaired electron on Cu is in a d orbital which has its maximum overlap with the F^- ions in the x direction and weak antiferromagnetic exchange along the y and z direction. Then, if the temperature is high enough (higher than the latter exchange interaction), the system behaves effectively as one-dimensional.

One more example is provided by the whole family of Bechgaard salts $(TMTSF)_2X$, which have been objects of intense study, as the first organic superconductors. They appear in crystalline form, in which piles of (TMTSF) are sided by PF_6^- (hexafluorophosphate) or ClO_4 (perchlorade) anions, which provide a negative charge and bind the salt together. The π orbitals of sulfur have a high overlap among them, hence allowing high conductivity along the direction of the cations. Conversely, due to the ionic nature of the compound, the transit of an electron in the perpendicular plane is highly suppressed. In particular, there is one ion for two (TMTSF) molecules, which implies that the chain is at quarter filling.

More recently, the Heisenberg chain has been applied to explain the results of neutron scattering experiments [16] on Azurite ($Cu_3(CO_3)_2(OH_2)$), with success in the prediction of the dynamical properties in a magnetic field. Other applications can be found in [17, 18].

1.3.1 Construction of the Hamiltonian

We start by rewriting the operator (1.11) as

$$L(\lambda)_{a,m} = \frac{f_1(\lambda)}{\sinh(\lambda + i\,\gamma/2)}\,\mathbf{1} + \frac{f_p(\lambda)}{\sinh(\lambda + i\,\gamma/2)} P_{a,m} + \frac{f_z(\lambda)}{\sinh(\lambda + i\,\gamma/2)} \sigma_a^z \sigma_m^z \tag{1.55}$$

with

$$\begin{aligned} f_1(\lambda) &= \frac{1}{2}\left(\sinh(\lambda + i\,\gamma/2) + \sinh(\lambda - i\,\gamma/2) - \sinh(i\,\gamma)\right) \\ f_P(\lambda) &= \sinh(i\,\gamma) \\ f_z(\lambda) &= \frac{1}{2}\left(\sinh(\lambda + i\,\gamma/2) - \sinh(\lambda - i\,\gamma/2) - \sinh(i\,\gamma)\right) \end{aligned}$$

In the homogenous case, the monodromy matrix is

$$\begin{aligned} T(\lambda) &= \frac{1}{\sinh(\lambda + i\,\gamma/2)^{2N}} \left(f_1\,\mathbf{1} + f_P P_{a,2N} + f_z \sigma_a^z \sigma_{2N}^z\right)\left(f_1\,\mathbf{1} + f_P P_{a,2N-1} + f_z \sigma_a^z \sigma_{2N-1}^z\right)\ldots \\ &\quad \ldots \left(f_1\,\mathbf{1} + f_P P_{a,1} + f_z \sigma_a^z \sigma_1^z\right) \end{aligned} \tag{1.56}$$

The transfer matrix is defined as usual as the trace over the auxiliary space of the previous expression. In particular, we have that

$$\tau(i\,\gamma/2) = \mathrm{Tr}_a\left[T(i\,\gamma/2)\right] = P_{1,2} P_{2,3} \ldots P_{2N-1,2N} = V \tag{1.57}$$

is the shift operator.

Being V a translation operator, a lattice momentum can be defined as its generator, as $V = e^{-iaP}$, with a the lattice spacing. It follows that the first integral of motion is simply

$$Q_0 = \log \tau(i\,\gamma/2) = -iaP \tag{1.58}$$

The second integral of motion has to be computed by using that

$$\begin{aligned} D_{a,m} = \frac{d}{d\lambda} L_{a,m}(i\,\gamma/2) &= \frac{\cosh^2(i\,\gamma/2)\,\mathbf{1} - \cosh(i\,\gamma) P_{a,m} - \sinh^2(i\,\gamma/2)\sigma_a^z \sigma_m^z}{\sinh(i\,\gamma)} \\ L_{a,m}(i\,\gamma/2) &= P_{a,m} \end{aligned}$$

Then the derivative of the transfer matrix is

$$\begin{aligned} \frac{d\hat{\tau}}{d\lambda}(i\,\gamma/2) &= \sum_j \mathrm{Tr}_a \left[P_{a,N} \dots P_{a,j+1} D_{a,j} P_{a,j-1} \dots P_{a,1} \right] \\ &= \sum_j D_{j-1,j} P_{N,N-1} \dots P_{j+2,j+1} P_{j+1,j-1} P_{j-1,j-2} \dots P_{2,1} \end{aligned} \tag{1.59}$$

and the logarithmic derivative is simply written as

$$\frac{d \log \hat{\tau}}{d\lambda}(i\,\gamma/2) = \sum_j D_{j-1,j} P_{j-1,j} \tag{1.60}$$

then one has

$$\begin{aligned} \frac{J}{2} \sinh(i\,\gamma) \frac{d \log \hat{\tau}}{d\lambda}(i\,\gamma/2) &= \frac{J}{2} \sum_{n=1}^{2N} \left[\cosh^2(i\,\gamma/2) P_{n,n+1} - \cosh(i\,\gamma)\,\mathbf{1} - \sinh^2(i\,\gamma/2)\sigma_n^z \sigma_{n+1}^z \right] \\ &= \frac{J}{4} \sum_{n=1}^{2N} \left[\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \cos\gamma\, \sigma_j^z \sigma_{j+1}^z \right] + const \end{aligned} \tag{1.61}$$

which is just (1.54), apart from a constant term.

It is instructive, now that we have (1.56), to try to understand the action of the $\mathcal{B}$ operators for this specific model. In order to do this, we can try to construct them explicitly, for a few-sites chain. Obviously, for one site only, the ladder operator just flips an up spin downwards. For a two-site chain, it reads:

$$\mathcal{B}(\lambda) = A_1(\lambda - h_1) B_2 - B_1 A_2(\lambda - h_2) \tag{1.62}$$

and for a three-site chain:

$$\mathcal{B}(\lambda) = A_1(\lambda - h_1) A_2(\lambda - h_2) B_3 + B_1 B_2 B_3 - A_1(\lambda - h_1) B_2 A_3(h_3 - \lambda) + B_1 A_2(h_2 - \lambda) A_3(h_3 - \lambda) \tag{1.63}$$

where the notation used is:

$$A_j(\lambda) = q^{-i\,\lambda} q^{S_j^3} - q^{i\,\lambda} q^{-S_j^3}\,, \qquad B_j = \left(q^{-i\,\lambda} - q^{i\,\lambda} \right) S_j^- \tag{1.64}$$

In principle, one can go on to higher and higher sizes, but the main feature of the operators is already clear: as anticipated by (1.53) the $\mathcal{B}$s act on a state by flipping downwards one spin, which is localized at a given site with a certain amplitude.

1.3.2 Scalar products and matrix elements

A beautiful formula, due to N. Slavnov [19, 13], is known for the scalar product of two states, provided that at least one of them is an eigenstate of the Hamiltonian of the XXZ chain:

$$\langle \psi(\{\mu\}) | \prod_{j=1}^{N} B(\lambda_j) | 0 \rangle = \frac{\prod_{a=1}^{M} d(\mu_a)}{\prod_{a>b}^{M} \sinh(\mu_a - \mu_b) \sinh(\lambda_b - \lambda_a)} \cdot \det H(\{\mu\}, \{\lambda\}) \tag{1.65}$$

in which H is

$$\begin{aligned} H^{\omega}_{jk} &= a(\lambda_j)\, t(\mu_k,\lambda_j) \prod_{l=1}^{M} \sinh(\mu_l - \lambda_j - i\gamma) \\ &\quad -e^{-2i\omega} d(\lambda_j)\, t(\lambda_j,\mu_k) \prod_{l=1}^{M} \sinh(\mu_l - \lambda_j + i\gamma) \end{aligned} \tag{1.66}$$

with $1 \leq j,k \leq M$, a, d defined in (1.44) and

$$t(\mu,\lambda) = \frac{-i\sin\gamma}{\sinh(\mu-\lambda)\sinh(\mu-\lambda-i\gamma)}. \tag{1.67}$$

We would like to emphasize that the rapidities of the "bra" set satisfy the Richardson equations and therefore are an eigenstate of the transfer matrix, while those appearing in the "ket" are just M complex numbers.

The quantum inverse problem (QIP) amounts to reconstructing the operators of the quantum system under study (the S^+, S^-, S^z operators on every site of the spin chain) in terms of the ABA operators $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$. Since these operators are used to build eigenstates and they form the algebra (1.26), the solution of the inverse problem [13, 14] opens the way to the computation of matrix elements and from these of expectation values and correlation functions. For the Heisenberg chain, the results are:

$$S^-_m = \left\{\prod_{l=1}^{m-1} \hat{\tau}(h_l)\right\} \mathcal{B}(h_m) \left\{\prod_{l=1}^{m} \hat{\tau}(h_l)\right\}^{-1} \tag{1.68}$$

$$S^+_m = \left\{\prod_{l=1}^{m-1} \hat{\tau}(h_l)\right\} \mathcal{C}(h_m) \left\{\prod_{l=1}^{m} \hat{\tau}(h_l)\right\}^{-1} \tag{1.69}$$

$$S^z_m = \left\{\prod_{l=1}^{m-1} \hat{\tau}(h_l)\right\} \mathrm{Tr}_a\left[T_a(h_m)\cdot S^z_a\right] \left\{\prod_{l=1}^{m} \hat{\tau}(h_l)\right\}^{-1} \tag{1.70}$$

Matrix elements of operators in the context of quantum spin chains are denominated "form factors". Throughout this thesis, we would like to reserve such a denomination to the corresponding objects in quantum field theory. As explained in Chapter 5, the two families have a close relationship in a lattice model of our interest: the inhomogeneous XXZ chain. Because of this, to avoid confusion, we will refer to the ones coming from the lattice simply as *matrix elements*, reserving the denomination of *form factors* to the corresponding objects in the content of integrable field theories.

Given this solution, it is easy to show that the matrix element of the site magnetization operator $F^{S^z_m}(\{\mu\},\{\lambda\}) = \langle\{\mu\}|\, S^z_m\, |\{\lambda\}\rangle$ between Bethe eigenstates can be represented as a determinant

$$F^{S^z_m}(\{\mu\},\{\lambda\}) = \frac{\phi_{m-1}(\{\mu\})}{\phi_{m-1}(\{\lambda\})} \prod_{l=1}^{m} \frac{\sinh(\mu_j - h_m + i\gamma)}{\sinh(\lambda_j - h_m + i\gamma)} \frac{\det\left[H - 2P_m\right]}{\prod_{j>k}\sinh(\mu_k-\mu_j)\sinh(\lambda_j-\lambda_k)} \tag{1.71}$$

where the matrix H has been given in (1.66) and P is the rank-one matrix:

$$(P_m)_{j,k} = t(\mu_j - h_m) \prod_{l=1}^{M} \sinh(\lambda_l - \lambda_h + i\gamma) \tag{1.72}$$

The coefficients $\phi_m(\{\lambda_k\})$ are instead:

$$\phi_m(\{\lambda_k\}) = \prod_{k=1}^{n} \prod_{j=1}^{m} b^{-1}(\lambda_k - \xi_j), \tag{1.73}$$

with b defined in 1.5.

The proof is given in [13] and uses the solution of the inverse problem (1.70) by writing

$$\sigma_m^z = 2 \prod_{j=1}^{m-1} (\mathcal{A}+\mathcal{D})(h_j) \cdot A(h_m) \cdot \prod_{j=m+1}^{N} (\mathcal{A}+\mathcal{D})(h_j) - I, \tag{1.74}$$

Ttherefore, taking the matrix element between two states, one hase

$$F_n^z(m,\{\mu\},\{\lambda\}) = 2\ \phi_m^{-1}(\{\lambda_k\})\, \phi_{m-1}(\{\mu_j\})\ P_1(h_m,\{\mu\},\{\lambda\}) - S(\{\mu\},\{\lambda\})$$

with

$$P_1(h_m,\{\mu\},\{\lambda\}) = \langle 0| \prod_{j=1}^{n} \mathcal{C}(\mu_j)\ \mathcal{A}(h_m) \prod_{k=1}^{n} \mathcal{B}(\lambda_k)\ |0\rangle\,. \tag{1.75}$$

To compute the function P_1, one uses the formula (1.47) for the action of the operator $A(h_m)$ on an arbitrary state:

$$\begin{aligned} \mathcal{A}(h_m) \prod_{k=1}^{n} \mathcal{B}(\lambda_k)\,|0\rangle \;&=\; \prod_{k=1}^{n} \frac{\sinh(\lambda_k - h_m - i\gamma)}{\sinh(\lambda_k - h_m)} \prod_{k=1}^{n} \mathcal{B}(\lambda_k)\,|0\rangle - \\ &\quad - \sum_{a=1}^{N} \frac{\sinh(i\gamma)}{\sinh(\lambda_a - h_m)} \left(\prod_{k\neq a}^{n} \frac{\sinh(\lambda_k - \lambda_a + i\gamma)}{\sinh(\lambda_k - \lambda_a)} \right) \mathcal{B}(h_m) \prod_{k\neq a}^{n} \mathcal{B}(\lambda_k)\,|0\rangle \end{aligned} \tag{1.76}$$

Hence P_1 reduces to a sum of scalar products, therefore to a sum of determinants according to (1.65). It can be rewritten as a single determinant by means of the following formula for the determinant of the sum of two matrices, one of which being of rank one. Indeed, if $\hat{A}$ is an arbitrary $n \times n$ matrix and $\hat{B}$ a rank one $n \times n$ matrix, the determinant of the sum $\hat{A} + \hat{B}$ is:

$$\det(\hat{A} + \hat{B}) = \det \hat{A} + \sum_{j=1}^{n} \det \hat{A}^{(j)},$$

where

$$\begin{aligned} \hat{A}^{(j)}_{ab} &= \hat{A}_{ab} \quad \text{for} \quad b \neq j, \\ \hat{A}^{(j)}_{aj} &= \hat{B}_{aj}. \end{aligned}$$

A determinant representation for the matrix elements

$$F_n^-(m,\{\mu\},\{\lambda\}) = \langle 0| \prod_{j=1}^{n+1} C(\mu_j)\ \sigma_m^- \prod_{k=1}^{n} B(\lambda_k)\ |0\rangle\,, \tag{1.77}$$

and

$$F_n^+(m,\{\mu\},\{\lambda\}) = \langle 0| \prod_{k=1}^{n} C(\lambda_k)\ \sigma_m^+ \prod_{j=1}^{n+1} B(\mu_j)\ |0\rangle\,, \tag{1.78}$$

where $\{\lambda_k\}_n$ and $\{\mu_j\}_{n+1}$ are solutions of Bethe equations, is also known. For two Bethe states with spectral parameters $\{\lambda_k\}_n$ and $\{\mu_j\}_{n+1}$, the matrix element of the operator σ_m^- can be represented as a determinant,

$$\begin{aligned} F_n^-(m,\{\mu\},\{\lambda\}) &= \frac{\phi_{m-1}(\{\mu_j\})}{\phi_{m-1}(\{\lambda_k\})}\frac{\prod_{j=1}^{n+1}\sinh(\mu_j-\xi_m+\eta)}{\prod_{k=1}^{n}\sinh(\lambda_k-\xi_m+\eta)}\times \\ &\times\frac{\det_{n+1}H^-(m,\{\mu\},\{\lambda\})}{\prod_{n+1\geq k>j\geq 1}\sinh(\mu_k-\mu_j)\prod_{1\leq\beta<\alpha\leq n}\sinh(\lambda_\beta-\lambda_\alpha)}, \end{aligned} \tag{1.79}$$

and the $(n+1)\times(n+1)$ matrix H^- is defined as

$$H^-_{ab}(m)=\frac{\sinh(\eta)}{\sinh(\mu_a-\lambda_b)}\left(a(\lambda_b)\prod_{j=1,j\neq a}^{n+1}\sinh(\mu_j-\lambda_b+\eta)-d(\lambda_b)\prod_{j=1,j\neq a}^{n+1}\sinh(\mu_j-\lambda_b-\eta)\right)$$
$$\text{for}\quad b<n+1, \tag{1.80}$$

$$H^-_{an+1}(m)=\frac{\sinh(\eta)}{\sinh(\mu_a-\xi_m+\eta)\sinh(\mu_a-\xi_m)}. \tag{1.81}$$

The matrix element $F_n^+(m,\{\mu\},\{\lambda\})$ of the operator σ_m^+ admits a similar representation,

$$F_n^+(m,\{\mu\},\{\lambda\})=\frac{\phi_m(\lambda_k)\,\phi_{m-1}(\lambda_k)}{\phi_{m-1}(\mu_j)\,\phi_m(\mu_j)}F_n^-(m,\{\mu\},\{\lambda\}). \tag{1.82}$$

The proof of these representations is rather straightforward using 1.68, as the local operator σ_m^- can be expressed in terms of the transfer matrix and the operator $B(\xi_m)$ as

$$\sigma_m^-=\prod_{j=1}^{m-1}(\mathcal{A}+\mathcal{D})(\xi_j)\ \cdot\ \mathcal{B}(\xi_m)\ \cdot\ \prod_{j=m+1}^{N}(\mathcal{A}+\mathcal{D})(\xi_j)$$

Since the Bethe states are eigenstates of the transfer matrix,

$$(A(\xi_j)+D(\xi_j))\prod_{k=1}^{n}B(\lambda_k)\,|0\rangle=\left(\prod_{a=1}^{n}b^{-1}(\lambda_a,\xi_j)\right)\prod_{k=1}^{n}B(\lambda_k)\,|0\rangle\,, \tag{1.83}$$

the product of the operators $A(\xi_j)+D(\xi_j)$ contributes to the function $F_n^-(m,\{\mu\},\{\lambda\})$ as a global factor:

$$F_n^-(m,\{\mu\},\{\lambda\})=\phi_m^{-1}(\{\lambda_k\})\,\phi_{m-1}(\{\mu_j\})\ \langle 0|\ \prod_{j=1}^{n+1}C(\mu_j)\ B(\xi_m)\prod_{k=1}^{n}B(\lambda_k)\ |0\rangle\,. \tag{1.84}$$

Here we used a simple property of the solutions of Bethe equations,

$$\prod_{k=1}^{n}\prod_{j=1}^{N}b^{-1}(\lambda_k,\xi_j)=1.$$

The right hand side of (1.84) thus reduces to a scalar product,

$$F_n^-(m,\{\mu\},\{\lambda\})=\phi_m^{-1}(\{\lambda_k\})\,\phi_{m-1}(\{\mu_j\})\ S_{n+1}(\{\mu_j\},\{\xi_m,\lambda_1,\ldots,\lambda_n\}), \tag{1.85}$$

which, $\{\mu_j\}$ being a solution of Bethe equations, can be computed by means of (1.65). Writing this explicitly, the representation (1.79) is obtained.

The form factor $F_n^+(m,\{\mu\},\{\lambda\})$ can be calculated analogously using the representation for the operator σ_m^+ given by 1.68.

1.4 The Richardson model

The Richardson model [20, 21, 22] belongs to the class of fully connected models. The Hamiltonian is written in terms of creation and destruction operators of fermions in energy levels $\alpha = 1, \ldots, N$ with spin $\uparrow$ or $\downarrow$ and reads:

$$H_R = \sum_{\alpha=1}^{N} h_\alpha \left(c^\dagger_{\alpha\uparrow} c_{\alpha\uparrow} + c^\dagger_{\alpha\downarrow} c_{\alpha\downarrow} \right) - g \sum_{\alpha\beta=1}^{N} c^\dagger_{\alpha\uparrow} c^\dagger_{\alpha\downarrow} c_{\beta\downarrow} c_{\beta\uparrow} \tag{1.86}$$

where the h_α are the single-particles energies of the N levels and g is a real (positive or negative) parameter, which models the matrix element of the scattering among Cooper pairs of spin-reversed electrons.

The Richardson model has been studied for long time in the context of nuclear superconductivity, where it was first applied [20, 21].More recently, it has been applied with great success to the study of the tunnelling spectra of metallic nanograins [23, 24, 25], where the spectroscopic gap between Al grains with an odd or even number of electrons was explained with the existence of pairing correlations among these [26]. Moreover, the model is important in connection to the finite-size scaling of the BCS theory of superconductivity [27, 28, 29, 30].

Justification for the use of (1.86) in condensed matter will be given in Section 2.1 in the context of BCS theory of superconductivity. To fully exploit the formalism presented for spin models, one can rephrase the Hamiltonian in terms of the $su(2)$ algebra generators:

$$H_R = -2 \sum_{\alpha=1}^{N} h_\alpha S^z_\alpha - g \sum_{\alpha\beta=1}^{N} S^+_\alpha S^-_\beta + \sum_{\alpha=1}^{N} h_\alpha \tag{1.87}$$

or equivalently in terms of Cooper pairs of fermions, i.e., of hardcore bosons:

$$H_R = 2 \sum_{\alpha=1}^{N} h_\alpha b^\dagger_\alpha b_\alpha - g \sum_{\alpha\beta=1}^{N} b^\dagger_\alpha b_\beta \tag{1.88}$$

under the definitions:

$$b^\dagger_\alpha = S^-_\alpha = c^\dagger_{\alpha\uparrow} c^\dagger_{\alpha\downarrow} \,, \quad b_\alpha = S^+_\alpha = c_{\alpha\downarrow} c_{\alpha\uparrow} \,, \quad 2S^z_\alpha = 1 - c^\dagger_{\alpha\uparrow} c_{\alpha\uparrow} - c^\dagger_{\alpha\downarrow} c_{\alpha\downarrow} \tag{1.89}$$

Note that the Hamiltonians (1.88,1.87) above are truly equivalent because, in addition to the formal mapping of the operators, there is also a one-to-one correspondence among the states in the Hilbert space of hard-core bosons and of spins, since:

$$|1\rangle \leftrightarrow |\downarrow\rangle \,, \qquad |0\rangle \leftrightarrow |\uparrow\rangle \tag{1.90}$$

On the other hand, the Hilbert spaces on which the operator (1.88) acts is only a subspace of the full space on which (1.86) acts. In facts, the states which are singly occupied, i.e., those in which there is only one electron with either $\uparrow$ spin or $\downarrow$ spin, are unaffected by the interaction and the net effect arising from their presence is that of "blocking" the level, by preventing the scattering of the other pairs on it. In other words, the full Hilbert space is divided into sectors with a given number of unpaired electrons and with given distribution over the levels. In each of these subspaces, the bosonic Hamiltonian (1.88) only describes the doubly occupied ("unblocked") levels, while the free part of (1.86) describes the remaining electrons.

1.4.1 Algebraic construction

Integrability of the model [31] was not realized immediately. Here we would like to emphasize the fact that it can be constructed from the general solution of the Yang-Baxter equation of section 1.2.1. An important property ("quasi-classical" property) of the matrix (1.10) is that it reduces to the identity in the limit:

$$\lim_{\eta\to 0} R(u) = \mathbf{1} \tag{1.91}$$

and because of this, it can be parametrized as:

$$R(u) = \mathbf{1} + \eta\tilde{R}(u) + O(\eta^2) \tag{1.92}$$

Here we follow the construction of [32]; the starting form of the monodromy matrix is the one given in (1.17), but here we chose to scale also the twist parameter with η as $\omega = \eta/g$. In other words, the Sklyanin matrix is $K_a = e^{-2i\eta/g(\sigma_a^z - 1)}$. Note that we consider the explicit form of the L-operator anticipated in (1.12).

Considering the reference state $|\uparrow, \ldots, \uparrow\rangle$ as above, we can compute the action of the monodromy matrix, yielding (1.45). From this, the Bethe equations are readily derived to be

$$\frac{a(w_j)}{d(w_j)} \prod_{k\neq j}^{M} \frac{w_j - w_k - \eta}{w_j - w_k + \eta} = 1 \tag{1.93}$$

as well as the eigenvalue of the transfer matrix on a Bethe state (1.39) defined by a set $\{w\}$ of rapidities:

$$\tau(u) = a(u) \prod_{j=1}^{M} \frac{u - w_j + \eta}{u - w_j} + e^{\frac{2\eta}{g}} d(u) \prod_{j=1}^{M} \frac{u - w_j - \eta}{u - w_j} \tag{1.94}$$

Once again, we define the transfer matrix according to (1.34), but in the following the construction of the Hamiltonian will slightly differ from the one for the Heisenberg chain. In facts, we can define $2N$ operators by:

$$\tilde{\tau}_\alpha = \lim_{\eta\to 0} \lim_{u\to h_\alpha} \frac{u - h_\alpha}{\eta^2} \hat{\tau}(u) \tag{1.95}$$

called *Gaudin Hamiltonians*. They can be written explicitly as:

$$\tilde{\tau}_\alpha = -\frac{2}{g} S_\alpha^z + \sum_{\beta\neq\alpha} \frac{2\vec{S}_\alpha \cdot \vec{S}_\beta}{h_\alpha - h_\beta} \tag{1.96}$$

and possess the key property of being mutually commuting:

$$[\tilde{\tau}_\alpha, \tilde{\tau}_\beta] = 0 \tag{1.97}$$

as derived from the commutativity of transfer matrices and the definition (1.95). We are in the position of defining an Hamiltonian through:

$$H_R = -g \sum_{\alpha=1}^{2N} h_\alpha \tilde{\tau}_\alpha + \frac{g^3}{4} \sum_{\alpha,\beta=1}^{2N} \tilde{\tau}_\alpha \tilde{\tau}_\beta + \frac{g^2}{2} \sum_{\alpha=1}^{2N} \tilde{\tau}_\alpha - g \sum_{\alpha=1}^{2N} \vec{S}_\alpha \cdot \vec{S}_\alpha \tag{1.98}$$

which produces exactly (1.87). Due to this construction, we have $2N$ integrals of motion (1.96) commuting among themselves and with the Hamiltonian. Note that this is so independently of the spin representation of $su(2)$, but throughout all this work, we are going to use the spin$-1/2$ representation.

Since the RTT-relation (1.23) must be satisfied order by order, then one also has:

$$T(u)_a = \mathbf{1} + \eta \tilde{T}(u) + O(\eta^2) \tag{1.99}$$

then the second order defines the *Gaudin algebra* (GA) much in the same way as for the YBA, since

$$\left[\tilde{T}_a(u), \tilde{T}_b(v)\right] = \left[\tilde{T}_a(u) + \tilde{T}_b(v), \tilde{R}_{a,b}(u-v)\right] \tag{1.100}$$

Again, writing the components of (1.99) in full:

$$\begin{aligned}
\mathcal{A}(u) &= \mathbf{1} + \eta\, \tilde{\mathrm{A}}(u) + O(\eta^2) && (1.101)\\
\mathcal{B}(u) &= \eta\, \tilde{\mathrm{B}}(u) + O(\eta^2) && (1.102)\\
\mathcal{C}(u) &= \eta\, \tilde{\mathrm{C}}(u) + O(\eta^2) && (1.103)\\
\mathcal{D}(u) &= \mathbf{1} + \eta\, \tilde{\mathrm{D}}(u) + O(\eta^2) && (1.104)
\end{aligned}$$

one obtains their commutation relations

$$\begin{aligned}
\left[\tilde{\mathrm{A}}(u), \tilde{\mathrm{A}}(v)\right] &= \left[\tilde{\mathrm{D}}(u), \tilde{\mathrm{D}}(v)\right] = 0 && (1.105)\\
\left[\tilde{\mathrm{B}}(u), \tilde{\mathrm{B}}(v)\right] &= \left[\tilde{\mathrm{C}}(u), \tilde{\mathrm{C}}(v)\right] = \left[\tilde{\mathrm{A}}(u), \tilde{\mathrm{D}}(v)\right] = 0 && (1.106)\\
\left[\tilde{\mathrm{B}}(u), \tilde{\mathrm{C}}(v)\right] &= \frac{\tilde{\mathrm{A}}(u) - \tilde{\mathrm{A}}(v) + \tilde{\mathrm{D}}(v) - \tilde{\mathrm{D}}(u)}{u - v} && (1.107)\\
\left[\tilde{\mathrm{A}}(u), \tilde{\mathrm{B}}(v)\right] &= \frac{\tilde{\mathrm{B}}(u) - \tilde{\mathrm{B}}(v)}{u - v} && (1.108)\\
\left[\tilde{\mathrm{A}}(u), \tilde{\mathrm{C}}(v)\right] &= \frac{\tilde{\mathrm{C}}(v) - \tilde{\mathrm{C}}(u)}{u - v} && (1.109)\\
\left[\tilde{\mathrm{D}}(u), \tilde{\mathrm{B}}(v)\right] &= \frac{\tilde{\mathrm{B}}(v) - \tilde{\mathrm{B}}(u)}{u - v} && (1.110)\\
\left[\tilde{\mathrm{D}}(u), \tilde{\mathrm{C}}(v)\right] &= \frac{\tilde{\mathrm{C}}(u) - \tilde{\mathrm{C}}(v)}{u - v} && (1.111)
\end{aligned}$$

The realization of the Gaudin-Yang-Baxter algebra associated with the monodromy matrix in the semiclassical limit can be written explicitly in a simple form:

$$\tilde{\mathrm{A}}(w) = -\frac{1}{g}\,\mathbf{1} + \sum_\alpha \frac{S^z}{w - h_\alpha} \tag{1.112}$$

$$\tilde{\mathrm{B}}(w) = \sum_\alpha \frac{S^-}{w - h_\alpha} \tag{1.113}$$

$$\tilde{\mathrm{C}}(w) = \sum_\alpha \frac{S^+}{w - h_\alpha} \tag{1.114}$$

$$\tilde{\mathrm{D}}(w) = \frac{1}{g}\,\mathbf{1} - \sum_\alpha \frac{S^z}{w - h_\alpha} \tag{1.115}$$

We are going to repeat now the construction of eigenstates of the transfer matrix to first order in the "quantum" parameter η exactly as in Section 1.2.2. The only difference is that we are using the Gaudin algebra to write the states:

$$|\{w\}\rangle = \tilde{\mathrm{B}}(w_1)\dots\tilde{\mathrm{B}}(w_M)\,|\uparrow\uparrow\dots\uparrow\rangle \tag{1.116}$$

Note that the generalized ladder operator $\tilde{\mathrm{B}}$ (1.113) flips a spin(creates a boson) on every level with an amplitude of probability which depends from its argument. The filling of the level α of a state which is created by the action of one operator (1.113) is just:

$$f_\alpha = \frac{1}{\sum_\beta \frac{|w-h_\alpha|^2}{|w-h_\beta|^2}} \tag{1.117}$$

Also the functions a, d defined from the eigenvalues of the transfer matrix on the reference state (1.43) are expanded in η:

$$a(u) = 1 + \eta\tilde{a}(u) + O(\eta^2)\,, \qquad d(u) = 1 + \eta\tilde{d}(u) + O(\eta^2) \tag{1.118}$$

and the rapidities $\{w_1,\dots,w_M\}$ of the eigenstates satisfy a system of equations which is derived from the rational model ones (1.93) system by expanding in η to first order:

$$\frac{1}{g} + \sum_{\alpha=1}^{N}\frac{1}{w_j - h_\alpha} - \sum_{k\neq j}^{M}\frac{2}{w_j - w_k} = 0 \qquad j = 1,\dots,M \tag{1.119}$$

Since first proposed by Richardson, these equations are are named after him. Analogously, all eigenvalues of the conserved operators (1.96) on a Bethe state $|\{w\}\rangle$ are obtained from (1.95) and (1.94) to be:

$$\lambda_\alpha = \frac{2}{g} + \sum_{\beta\neq\alpha}\frac{1}{h_\beta - h_\alpha} - \sum_{j=1}^{M}\frac{2}{h_\alpha - w_j} \tag{1.120}$$

By substituting the eigenvalues (1.120) into the Hamiltonian (1.98), it is possible to write the eigenenergy of a state (1.116) as:

$$E_{\{w\}} = 2\sum_{j=1}^{M} w_j - \sum_{\alpha=1}^{2N} h_\alpha \tag{1.121}$$

1.4.2 Matrix elements

The rational limit of the formula (1.65) allows to treat the Richardson model and the isotropic Heisenberg chain:

$$\langle\{w\}|\prod_j \mathcal{B}(v_j)\,|ref\rangle = \frac{\prod_{j=1}^{M} d(w_j)}{\prod_{j>k}(w_j - w_k)(v_k - v_j)}\det F \tag{1.122}$$

The matrix appearing above is now given by:

$$H_{j,k} = \frac{\eta a(v_j)\prod_{l=1}^{M}(w_l - v_j + \eta)}{(w_k - v_j)(w_k - v_j + \eta)} - \frac{\eta d(v_j)\prod_{l=1}^{M}(w_l - v_j - \eta)}{(w_k - v_j)(w_k - v_j - \eta)} \tag{1.123}$$

where the functions a, d refer to the rational case (1.45).

$$\langle\Psi(\{w\})|\prod\tilde{\mathrm{B}}(v_j)\,|\uparrow,\ldots,\uparrow\rangle=\frac{\prod_{j=1}^{M}d(w_j)}{\prod_{j>k}(w_j-w_k)(v_k-v_j)}\det\tilde{H}\tag{1.124}$$

where by $\Psi(\{w\})$ we want to emphasize that the rapidities of the "bra" set satisfy the Richardson equations and therefore are an eigenstate of the transfer matrix, while those appearing in the "ket" are just M complex numbers. The matrix $\tilde{H}$ appearing above is obtained from (1.123) in the quasi-classical limit:

$$\tilde{H}_{j,k}=\lim_{\eta\to 0}\frac{1}{\eta^2}H_{j,k}=\frac{\prod_{l=1}^{M}(w_l-v_k)}{(w_j-v_k)^2}\left(\frac{1}{g}-\sum_{\alpha=1}^{2N}\frac{1}{v_k-h_\alpha}+\sum_{l\neq j}\frac{2}{v_k-w_l}\right)\tag{1.125}$$

An important case of the above formula, necessary for extracting numerical values for observables, is the one $\{w\}=\{v\}$, for which the Slavnov formula yields the norm of the Bethe state:

$$\langle\{w\}|\{w\}\rangle=\det\mathcal{K}\tag{1.126}$$

where

$$\mathcal{K}_{j,k}=\begin{cases}\tilde{a}'(w_j)-\tilde{d}'(w_j)-\sum_{l\neq j}^{M}\frac{2}{(w_j-w_l)^2}=\sum_{\alpha=1}^{2N}\frac{1}{(w_j-h_\alpha)^2}-\sum_{l\neq j}^{M}\frac{2}{(w_j-w_l)^2} & j=k\\ \frac{2}{(w_j-w_k)^2} & j\neq k\end{cases}\tag{1.127}$$

To compute the matrix elements for S^z_α, one uses that $\tilde{\mathrm{D}}$ has simple poles when the argument assumes the values of one of the levels h_α. The derivation goes like for the XXZ case, or one can directly make use of (1.71) in the rational case, then consider the quasi-classical limit, so that:

$$\begin{aligned}\langle\{w\}|\,S^z_\alpha\,|\{v\}\rangle &= -\lim_{u\to h_\alpha}(u-h_\alpha)\,\langle\{w\}|\,\tilde{\mathrm{D}}(u)\,|\{v\}\rangle\\ &= -\prod_{l=1}^{M}\frac{w_l-h_\alpha}{v_l-h_\alpha}\frac{1}{\prod_{k>j}^{M}(v_k-v_j)(w_j-w_k)}\det\left[\tilde{H}-2\tilde{P}_\alpha\right]\end{aligned}\tag{1.128}$$

The rank-one matrix $\tilde{P}$ is obtained from (1.72) by applying the usual procedure of analytic continuation $\gamma\to i\gamma$, the rational limit and the quasi-classical limit:

$$\left[\tilde{P}_\alpha\right]_{j,k}=\lim_{\eta\to 0}\frac{1}{\eta^2}\left[P_\alpha\right]_{j,k}=\frac{\prod_{l\neq k}(v_l-v_k)}{(w_j-h_\alpha)}\tag{1.129}$$

We will also need the action of S^+, S^- between two states, which can be obtained either from (1.79) the or by using the explicit form of $\tilde{\mathrm{B}}$, $\tilde{\mathrm{D}}$ as follows:

$$\begin{aligned}\langle\{v\}|\,S^-_\alpha\,|\{w\}\rangle=\langle\{w\}|\,S^+_\alpha\,|\{v\}\rangle &= \lim_{u\to h_\alpha}(u-h_\alpha)\,\langle\{w\}|\{v\},u\rangle\\ &= \frac{\prod_{l=1}^{M}(w_l-h_\alpha)}{\prod_{l=1}^{M-1}(v_l-h_\alpha)}\frac{\det\tilde{H}^-}{\prod_{j<k}(v_k-v_j)\,(w_j-w_k)}\end{aligned}\tag{1.130}$$

where

$$\tilde{H}^-_{j,k}=\begin{cases}\tilde{H}_{j,k} & k<M\\ \frac{1}{(w_j-h_\alpha)^2} & k=M\end{cases}\tag{1.131}$$

c

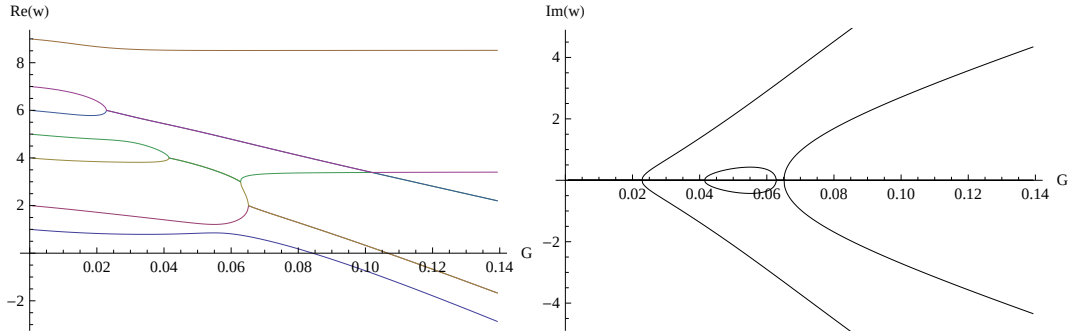


Figure 1.2: Real and imaginary parts of the roots of Richardson equations for an excited state with $N = 12$, $M = 7$, as g is varied. Roots collide and form a complex conjugate pair, or recombine and split. When g grows, they can go to $-\infty$ alone or in pairs, or they can stay finite, "trapped" between two adjacent levels.

1.4.3 Properties of Bethe states

The eigenstates are constructed like in (1.39), but with the use of the operator $\tilde{B}$ in (1.113) and rapidities satisfying (1.119).
This system of equations have a simple analogy in classical electrostatics: they left-hand side is the force acting on a mobile particle ("pairons") with unit charge at position w_j, as produced by the electric field generated by a set of sources ("orbitons") with double charge, fixed on the real axis at positions h_α. Moreover, external field, whose value is $1/g$, is present. The set of equations (1.119) express then equilibrium condition for the mobile charges. This is shown in Figure 1.1

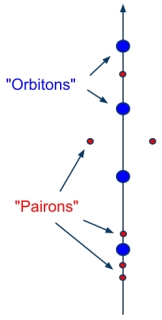


Figure 1.1: The electrostatic analogy of the Richardson equations

With this picture in mind, it is easy to figure out that mobile charges may either be found in between two fixed charges (real roots) or in pairs, symmetrically disposed on the two sides of the real axis (complex conjugated roots).

Solutions of the Richardson equations are easily guessed in the limit $g \to 0$, when the Hamiltonian just accounts for independent hardcore bosons distributed on different levels. In facts, the divergence from the first term must be compensated by an opposite one from the second term, which tells us that the solutions are all real and lie slightly below ($O(g)$) one of the energy levels. The full Hilbert space is then obtained by the 2^N configurations in which M roots are assigned to levels for each $M = 0, 1, \ldots, N$.

In the opposite limit $g \to \infty$, the Hamiltonian becomes

$$H_{R,g\to\infty} \simeq -g\left(\vec{S}\cdot\vec{S} - (S^z)^2 - S^z\right) \tag{1.132}$$

and conserves the total spin of the state and its z-projection. Numerical solutions shows that rapidities can either diverge to $-\infty$ proportionally to g or remain finite, with real part which

stays between two energy levels; consequently, the Richardson equations assume a simplified form [33] . For the r diverging roots, energy levels and finite roots can be neglected in a first approximation, and (1.119) become:

$$\frac{1}{g} + \frac{N}{w_j} - \frac{2(M-r)}{w_j} = \sum_{k \neq j}^{r} \frac{2}{w_j - w_k} \tag{1.133}$$

Instead, for the remaining $M - r$ roots, one has:

$$\sum_{\alpha=1}^{N} \frac{1}{w_j - h_\alpha} = 0 \tag{1.134}$$

If one multiplies each of the (1.133) by the corresponding w_j and adds all of them, then the leading part of the energy can be computed as:

$$E \simeq -gr(N - 2M + r + 1) \tag{1.135}$$

by comparing this expression with the strong coupling Hamiltonian (1.132) the number r of roots that diverge in the strong coupling limit and the total spin of a state are related [33] by:

$$r = s + M - \frac{N}{2} \tag{1.136}$$

An algorithm which can follow the evolution of the roots with g has to take into account these changes in the nature of the solution, where the roots become complex conjugate. These critical points, for random choices of the h's can occur at particularly close values of g and this can create troubles for the algorithm.[4] In order to pass the critical points a change of variable is needed, and one natural choice is [34]:

$$\begin{aligned} w_+ &= 2h_c - w_1 - w_2 \\ w_- &= (w_1 - w_2)^2 \end{aligned}$$

in which w_1 and w_2 are the root colliding on the level h_c.

When more than a pair of roots collide in a too small interval of g this change of variables may not be sufficiently accurate and one should think of something else (if one does not want to reduce the step in the increment of g indefinitely). The most general change of variables which smooths out the evolution across critical points is that which goes from the roots w_j to the coefficients c_i of the characteristic polynomial $p(w)$ –i.e. the polynomial whose all and only roots are the w_j's.

This polynomial is quite interesting in itself as it satisfies a second order differential equation whose polynomial solutions have been classified by Heines and Stjielties [35].[5] Following the evolution of the coefficients $c_i(g)$ is a viable alternative to following the roots but we found out that the best strategy is a combination of both evolutions. Therefore we follow the

[4] This problem is not so serious for the ground state and first excited states so one can go to much higher values of N without losing accuracy.

[5] The equation is $-h(x)p''(x) + \left(\frac{h(x)}{g} + h'(x)\right)p'(x) - V(x)p(x) = 0$, where $h(x) = \prod_{\alpha=1}^{N}(x - h_\alpha)$, $V(x) = \sum_{\alpha=1}^{N} \frac{h(x)A_\alpha}{x - h_\alpha}$. The problem to be solved is to find a set of A_α's such that there exists a polynomial solution of this equation. The solution will automatically satisfy also $A_\alpha = \frac{p'(h_\alpha)}{p(h_\alpha)}$.. For a more general method based on similar approach, see [36]

evolution of the roots, extrapolating the coefficients and using them to correct the position of the roots at the next step in the evolution. In this way we do not implement any change of variables explicitly and we do not have to track the position of critical points. This algorithm, implemented in Python, can be used on a desktop computer to find the roots of typical states with about 50 spins.

Part I

Integrability on the lattice

Chapter 2

Coupled Richardson models

This chapter approaches the study of the Josephson effect, with particular reference to the BCS-BEC crossover, through the few-body approximation constituted by the Richardson model.

Important technological applications of this effect are already widely used. For instance, superconducting quantum interference devices (SQUIDs), very sensitive magnetometers that operate via the Josephson effect, superconducting single-electron transistors (SSETs), circuit components constructed of superconducting materials making use of the Josephson effect to achieve novel properties, and rapid single flux quantum (RSFQ), a digital electronics technology that relies on Josephson junctions to process digital signals.

As it will be seen in this chapter, the Richardson model captures the essential features of the BCS theory of superconductivity, with the important extra feature of being exactly solvable. Moreover, it gives access to the few-body physics of superconducting devices, which has proven to play an important role in nanoscopic physics and may be relevant in issues related to miniaturization.

A review part, containing essential information on the BCS superconductivity, on the Josephson effect and on the BCS-BEC crossover, is contained in the first section. Our new results are presented in section 2.4. A warning for the reader is that here we will label the inhomogeneities in the construction of the transfer matrix as ϵ_α, since we are going to use them as single–particle energies (unlike in Chapter 3): all the formulas of Chapter 1 hold, provided one substitutes $h_\alpha \to \epsilon_\alpha$.

2.1 The BCS theory of superconductivity

Conduction electrons, residing in a crystal lattice, may be subject to an effective *attractive* interaction, resulting from the coupling with the lattice phonons [37, 38]. Although weak, this interaction is enough to bind quasiparticles of the Fermi liquid together [39] into *Cooper pairs*, with a finite binding energy $2\,\Delta_{\mathrm{BCS}}$. As a result, couples of fermions behave as *bosons*, that at low temperatures can form a Bose condensate.

Let us consider, in the grand-canonical ensemble, the many-body Hamiltonian obtained by a generic density-density interaction:

$$H - \mu N = \sum_{\mathbf{k}\sigma} (\varepsilon_{\mathbf{k}} - \mu)\, c^\dagger_{\mathbf{k},\sigma} c_{\mathbf{k},\sigma} - \sum_{\sigma_1,\ldots,\sigma_4} \sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_3+\mathbf{k}_4} V_{\mathbf{k}_1,\mathbf{k}_2;\mathbf{k}_3,\mathbf{k}_4} c^\dagger_{\mathbf{k}_1,\sigma_1} c^\dagger_{\mathbf{k}_2,\sigma_2} c_{\mathbf{k}_3,\sigma_3} c_{\mathbf{k}_4,\sigma_4} \quad (2.1)$$

where N is the total number operator and the $c_{\mathbf{k},\sigma}$, $c^\dagger_{\mathbf{k},\sigma}$ are fermionic operators that destroy

or create an electron with momentum $\mathbf{k}$ and spin orientation $\sigma =\uparrow,\downarrow= +,-$. They obey the anticommutation rule

$$\left\{c_{\mathbf{k},\sigma},\, c^{\dagger}_{\mathbf{k}',\sigma'}\right\} = \delta_{\mathbf{k},\mathbf{k}'}\delta_{\sigma,\sigma'} \tag{2.2}$$

Moreover, we have the matrix element

$$V_{\mathbf{k}_1,\mathbf{k}_2;\mathbf{k}_3,\mathbf{k}_4} = \frac{1}{2}\int dx_1 \int dx_2 \phi_{\mathbf{k}_1}(x_1)^* \phi_{\mathbf{k}_2}(x_2)^* V(x_1-x_2)\phi_{\mathbf{k}_2}(x_3)\phi_{\mathbf{k}_1}(x_4) \tag{2.3}$$

describing the momentum-conserving scattering of pairs of electrons among levels. If we assume that the dependence on momenta of the matrix element of the potential is weak, we can approximate $\langle \mathbf{k}_1\mathbf{k}_2|\, V\, |\mathbf{k}_3\mathbf{k}_4\rangle \simeq g$ which corresponds to a potential of the form $V(x) \sim g\delta(x)$, i.e., a point contact interaction. The resulting Hamiltonian is

$$H-\mu N = \sum_{\mathbf{k}\sigma}(\varepsilon_{\mathbf{k}}-\mu)\, c^{\dagger}_{\mathbf{k},\sigma}c_{\mathbf{k},\sigma} - \frac{g}{\Omega}\sum_{\sigma_1,\sigma_2}\sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_3+\mathbf{k}_4} c^{\dagger}_{\mathbf{k}_1,\sigma_1}c^{\dagger}_{\mathbf{k}_2,\sigma_2}a_{\mathbf{k}_3,\sigma_2}a_{\mathbf{k}_4,\sigma_1} \tag{2.4}$$

with Ω the volume (or any other appropriate normalization). We will incorporate this factor into the coupling constant, in the following.

To treat this Hamiltonian, it is convenient to define the following canonical transformation [40]:

$$c_{\mathbf{k},\uparrow} = u_{\mathbf{k}}A_{\mathbf{k}} + v^*_{\mathbf{k}}B^{\dagger}_{-\mathbf{k}}\,, \qquad c_{\mathbf{k},\downarrow} = u_{\mathbf{k}}B_{\mathbf{k}} - v^*_{\mathbf{k}}A^{\dagger}_{-\mathbf{k}} \tag{2.5}$$

with the constraint

$$|u_{\mathbf{k}}|^2 + |v_{\mathbf{k}}|^2 = 1 \tag{2.6}$$

necessary for the new quasiparticle operators to satisfy the anticommutation relations (2.2). The Bogolubov transform above may be inverted to give

$$A_{\mathbf{k},\sigma} = u_{\mathbf{k}}c_{\mathbf{k}\uparrow} - \sigma v_{\mathbf{k}}c^{\dagger}_{-\mathbf{k},\downarrow}\,, \qquad B_{\mathbf{k},\sigma} = u_{\mathbf{k}}c_{\mathbf{k}\downarrow} + \sigma v_{\mathbf{k}}c^{\dagger}_{-\mathbf{k},\uparrow} \tag{2.7}$$

The diagonal part of (2.1) is readily written, on the pseudovacuum, as

$$\sum_{\mathbf{k},\sigma}(\epsilon_{\mathbf{k}}-\mu)c^{\dagger}_{\mathbf{k},\sigma}c_{\mathbf{k},\sigma} = \sum_{\mathbf{k},\sigma}(\epsilon_{\mathbf{k}}-\mu)\left[2v^2_{\mathbf{k}} + (u^2_{\mathbf{k}}-v^2_{\mathbf{k}})(A^{\dagger}_{\mathbf{k}}A_{\mathbf{k}} + B^{\dagger}_{\mathbf{k}}B_{\mathbf{k}}) + 2u_{\mathbf{k}}v_{\mathbf{k}}(A^{\dagger}_{\mathbf{k}}B^{\dagger}_{-\mathbf{k}} + B_{-\mathbf{k}}A_{\mathbf{k}})\right] \tag{2.8}$$

The attractive interaction can make the Fermi sea unstable [41]; as a consequence, the quasiparticle operators above play an important role, since they defines an associated pseudovacuum in a natural way as:

$$A_{\mathbf{k}}\,|v\rangle = 0 \tag{2.9}$$

Let us consider a simple expectation value on the newly–defined state

$$\left\langle c^{\dagger}_{\mathbf{k},\sigma}c_{\mathbf{q},\sigma}\right\rangle = \delta_{\mathbf{k},\mathbf{q}}v^2_{\mathbf{k}} = \delta_{\mathbf{k},\mathbf{q}}n_{\mathbf{k},\sigma} \tag{2.10}$$

which shows how the electronic occupation number is related to the Bogolubov transform.

The interaction part can be separated into two contributions, according to the spin orientation of the scattered pairs which read:

$$\begin{aligned} V_a &= -\frac{1}{2}\sum_{\mathbf{k}\mathbf{k}'\mathbf{q}} V_{\mathbf{k},\mathbf{k}';\mathbf{k}+\mathbf{q},\mathbf{k}'-\mathbf{q}}\sum_{\sigma} c^{\dagger}_{\mathbf{k},\sigma}c^{\dagger}_{\mathbf{k}',\sigma}c_{\mathbf{k}'-\mathbf{q},\sigma}c_{\mathbf{k}+\mathbf{q},\sigma} \\ V_b &= -\sum_{\mathbf{k}\mathbf{k}'\mathbf{q}} V_{\mathbf{k},-\mathbf{k}';\mathbf{k}+\mathbf{q},-\mathbf{k}'-\mathbf{q}}c^{\dagger}_{\mathbf{k},\uparrow}c^{\dagger}_{\mathbf{k}',\downarrow}c_{\mathbf{k}'-\mathbf{q},\downarrow}c_{\mathbf{k}+\mathbf{q},\uparrow} \end{aligned} \tag{2.11}$$

By substituting the definition (2.5), they are brought into the form:

$$\begin{aligned}
V_a &= N(V_a) - \sum_{\mathbf{k},\mathbf{k}'} \left(V_{\mathbf{k},\mathbf{k}';\mathbf{k},\mathbf{k}'} - V_{\mathbf{k},\mathbf{k}';\mathbf{k}',\mathbf{k}}\right) [v_{\mathbf{k}}^2 v_{\mathbf{k}'}^2 + v_{\mathbf{k}'}^2(u_{\mathbf{k}}^2 - v_{\mathbf{k}}^2)(A_{\mathbf{k}}^\dagger A_{\mathbf{k}} + B_{-\mathbf{k}}^\dagger B_{-\mathbf{k}}) \\
&\quad + 2v_{\mathbf{k}'}^2 u_{\mathbf{k}} v_{\mathbf{k}}(A_{\mathbf{k}}^\dagger B_{-\mathbf{k}}^\dagger + B_{-\mathbf{k}} A_{\mathbf{k}})] \\
V_b &= N(V_b) - \sum_{\mathbf{k},\mathbf{k}'} (V_{\mathbf{k},-\mathbf{k};\mathbf{k}',-\mathbf{k}'} u_{\mathbf{k}} v_{\mathbf{k}} u_{\mathbf{k}'} v_{\mathbf{k}'} + V_{\mathbf{k},-\mathbf{k}';\mathbf{k},-\mathbf{k}'} v_{\mathbf{k}}^2 v_{\mathbf{k}'}^2) \\
&\quad - \sum_{\mathbf{k},\mathbf{k}'} [V_{\mathbf{k},-\mathbf{k}';\mathbf{k},-\mathbf{k}'}(u_{\mathbf{k}}^2 - v_{\mathbf{k}}^2) v_{\mathbf{k}'}^2 - 2V_{\mathbf{k},-\mathbf{k};\mathbf{k}',-\mathbf{k}'} u_{\mathbf{k}} v_{\mathbf{k}} u_{\mathbf{k}'} v_{\mathbf{k}'}](A_{\mathbf{k}}^\dagger A_{\mathbf{k}} + B_{-\mathbf{k}}^\dagger B_{-\mathbf{k}}) \\
&\quad - \sum_{\mathbf{k},\mathbf{k}'} [V_{\mathbf{k},-\mathbf{k};\mathbf{k}',-\mathbf{k}'}(u_{\mathbf{k}}^2 - v_{\mathbf{k}}^2) u_{\mathbf{k}'} v_{\mathbf{k}'} + 2V_{\mathbf{k},-\mathbf{k}';\mathbf{k},-\mathbf{k}'} u_{\mathbf{k}} v_{\mathbf{k}} v_{\mathbf{k}'}^2](A_{\mathbf{k}}^\dagger B_{-\mathbf{k}}^\dagger + B_{-\mathbf{k}} A_{\mathbf{k}})]
\end{aligned} \tag{2.12}$$

where $N(\mathcal{O})$ denotes the normal ordered form of the operator with respect to the state (1.40), which does not contribute when contracted on it. We now define a new single-particle energy $\epsilon_{\mathbf{k}} = \epsilon_{\mathbf{k}} - \sum_{\mathbf{k}'} V_{\mathbf{k},\mathbf{k};\mathbf{k}',\mathbf{k}'} v_{\mathbf{k}'}^2$ and measure it from the chemical potential $\xi_{\mathbf{k}} = \epsilon_{\mathbf{k}} - \mu$. We also introduce the *energy gap* by the relation

$$\Delta_{\mathbf{k}} = \sum_{\mathbf{k}'} V_{\mathbf{k},-\mathbf{k};\mathbf{k}',-\mathbf{k}'} u_{\mathbf{k}'} v_{\mathbf{k}'} \tag{2.13}$$

Then, following [42], it is possible to collect together all the similar contribution and to rewrite the Hamiltonian as:

$$K = U + H_1 + H_2 \tag{2.14}$$

where

$$\begin{aligned}
U &= 2\sum_{\mathbf{k}} \xi_{\mathbf{k}} v_{\mathbf{k}}^2 - \sum_{\mathbf{k}} u_{\mathbf{k}} v_{\mathbf{k}} \Delta_{\mathbf{k}} + \sum_{\mathbf{k}\mathbf{k}'} (V_{\mathbf{k},\mathbf{k}';\mathbf{k},\mathbf{k}'} - V_{\mathbf{k},\mathbf{k}';\mathbf{k}',\mathbf{k}} + V_{\mathbf{k},-\mathbf{k}';\mathbf{k},-\mathbf{k}'}) v_{\mathbf{k}}^2 v_{\mathbf{k}'}^2 \\
&\quad - \sum_{\mathbf{k}\mathbf{k}'} V_{\mathbf{k},-\mathbf{k};\mathbf{k}',-\mathbf{k}'} u_{\mathbf{k}} v_{\mathbf{k}} v_{\mathbf{k}'} v_{\mathbf{k}'}
\end{aligned} \tag{2.15}$$

$$H_1 = \sum_{\mathbf{k}} [(u_{\mathbf{k}}^2 - v_{\mathbf{k}}^2)\xi_{\mathbf{k}} + 2u_{\mathbf{k}} v_{\mathbf{k}} \Delta_{\mathbf{k}}](A_{\mathbf{k}}^\dagger A_{\mathbf{k}} + B_{-\mathbf{k}}^\dagger B_{-\mathbf{k}}) \tag{2.16}$$

$$H_2 = \sum_{\mathbf{k}} [2u_{\mathbf{k}} v_{\mathbf{k}} \xi_{\mathbf{k}} - (u_{\mathbf{k}}^2 - v_{\mathbf{k}}^2)\Delta_{\mathbf{k}}](A_{\mathbf{k}}^\dagger B_{-\mathbf{k}}^\dagger + B_{-\mathbf{k}} A_{\mathbf{k}}) \tag{2.17}$$

Here, we see why the choice (2.5) is successful in dealing with the pairing interaction in (2.1): it is possible to impose the constraint

$$2\xi_{\mathbf{k}} u_{\mathbf{k}} v_{\mathbf{k}} = (u_{\mathbf{k}}^2 - v_{\mathbf{k}}^2)\Delta_{\mathbf{k}} \tag{2.18}$$

to make the part which is not diagonal in the new operators vanish. The above equation, together with (2.6), has the solution

$$u_{\mathbf{k}}^2 = \frac{1}{2}\left(1 + \frac{\xi_{\mathbf{k}}}{\sqrt{\xi_{\mathbf{k}}^2 + \Delta_{\mathbf{k}}^2}}\right), \qquad v_{\mathbf{k}}^2 = \frac{1}{2}\left(1 - \frac{\xi_{\mathbf{k}}}{\sqrt{\xi_{\mathbf{k}}^2 + \Delta_{\mathbf{k}}^2}}\right) \tag{2.19}$$

while the constraint becomes

$$2u_{\mathbf{k}} v_{\mathbf{k}} = \Delta_{\mathbf{k}} / \sqrt{\xi_{\mathbf{k}}^2 + \Delta_{\mathbf{k}}^2} \tag{2.20}$$

and, together with (2.13), defines the *BCS gap equation*:

$$\Delta_{\mathbf{k}} = \frac{1}{2}\sum_{\mathbf{k}'} V_{\mathbf{k},-\mathbf{k}';\mathbf{k},-\mathbf{k}'} \frac{\Delta_{\mathbf{k}'}}{\sqrt{\xi_{\mathbf{k}}^2 + \Delta_{\mathbf{k}}^2}} \tag{2.21}$$

for the unknown function $\Delta_{\mathbf{k}}$. Whenever the two–body potential allows a nontrivial solution Δ_{BCS} of this equation, the latter is called a *superconducting* solution.

Then the Hamiltonian becomes:

$$K = U + \sum_{\mathbf{k}} \sqrt{\xi_{\mathbf{k}}^2 + {\Delta_{\mathrm{BCS}}}^2}(A_{\mathbf{k}}^\dagger A_{\mathbf{k}} + B_{-\mathbf{k}}^\dagger B_{-\mathbf{k}}) \tag{2.22}$$

which shows that the state (2.9) is indeed a vacuum state, above which positive–energy excitations (pseudoparticles) can be created. They have energy $\sqrt{\xi_{\mathbf{k}}^2 + {\Delta_{\mathrm{BCS}}}^2}$, so that there is a gap in the spectrum whenever this equation has a nonzero solution.

Note that the expectation value of the total number of electrons on the vacuum, according to (2.10) and (2.19), is given by

$$n = \sum_{\mathbf{k}} \langle n_{\mathbf{k},+} + n_{\mathbf{k},-} \rangle = \sum_{\mathbf{k}} 2v_{\mathbf{k}}^2 = \sum_{\mathbf{k}} \left(1 - \frac{\xi_{\mathbf{k}}}{\sqrt{\xi_{\mathbf{k}}^2 + {\Delta_{\mathrm{BCS}}}^2}} \right) \tag{2.23}$$

taking the name of *number equation*.

A great simplification is made if we assume that the matrix elements are constant in some region around the Fermi surface and vanish elsewhere:

$$V_{\mathbf{k},-\mathbf{k}';\mathbf{q},-\mathbf{q}} \simeq g\Theta(\omega - |\xi_{\mathbf{k}}|)\Theta(\omega - |\xi_{\mathbf{q}}|) \tag{2.24}$$

which is applicable when the interaction is originated by the scattering of phonons, as in the BCS theory [43]. In this case the gap is indeed independent from the level. If we write our pairing Hamiltonian only for the levels within this region (the others are free) within the approximation (2.24) and we relax the constraint of conservation of momenta, we obtain a mean–field approximation or *reduced BCS model*, being nothing else but (1.86).

Substitution of (2.19) into the expectation value (2.15) of the grand–canonical Hamiltonian on the new vacuum (2.9) yields the ground state energy:

$$\frac{E_0}{V} = \sum_{\mathbf{k}} \left(\epsilon_{\mathbf{k}} - \frac{\epsilon_{\mathbf{k}}\,\xi_{\mathbf{k}}}{\sqrt{\xi_{\mathbf{k}}^2 + {\Delta_{\mathrm{BCS}}}^2}} - \frac{1}{2}\frac{{\Delta_{\mathrm{BCS}}}^2}{\sqrt{\xi_{\mathbf{k}}^2 + {\Delta_{\mathrm{BCS}}}^2}} \right) \tag{2.25}$$

These equations are correctly reproduced by the Richardson model in the thermodynamic limit, as will be shown in the next section.

2.1.1 Large-N limit of the Richardson model

The Richardson model (1.86) arises from (2.4) by loosening the requirement of conservation of momentum in the interaction. Using the behaviour of solutions described in section (1.4.3) in the thermodynamic limit, it is possible to show that the Richardson model reproduces the BCS theory of superconductivity [44, 28, 29]. We remind the reader that in the ground state of a large system in which an even number of roots (particles) is present, all the roots

come in complex conjugated pairs, provided the pair scattering g is large enough. These pairs distribute in such a way to form an arch Γ in the complex plane, with extremes at the points:

$$\mu \pm i\,\Delta_{\rm BCS} \tag{2.26}$$

The thermodynamic limit we are interested in is obtained by letting the number of energy levels go to infinity in such a way that their range $\Omega = [-\omega/2, \omega/2]$ is kept constant, in a way that mimics a Debye energy. In other words, the energy spacing decreases like $d \sim 1/N$. Moreover, we should always consider a fixed filling x.

$$N \to \infty\,, \quad M \to \infty\,, \quad M/N = x\,, \quad g \to 0\,, \quad G = g\,N \tag{2.27}$$

The energy levels will therefore be most conveniently described by a density $\rho(\epsilon)$ of negative charge, located on the real axis. The total charge in the interval will be given by

$$N = 2\int_\Omega \rho(\epsilon) d\epsilon \tag{2.28}$$

in which the factor two comes from the fact that the "pairons" have double charge with respect to the "orbitons". The total number of pairs and the total energy are fixed by:

$$2M = \int_\Gamma r(w)|d\,w| \tag{2.29}$$

$$E = \int_\Gamma w r(w)|d\,w| \tag{2.30}$$

whereas equations (1.119) become

$$2\int \frac{\rho(\epsilon)}{w-\epsilon} d\epsilon - 2P\int \frac{r(v)}{w-v}|dv| + \frac{1}{G} = 0\,, \qquad w \in \Gamma \tag{2.31}$$

We imagine to start from the case in which we have a finite number of levels and we look for an analytic field $H(w)$ outside the curves Γ and Ω in the complex plane, in such a way that the poles of the field are in the position of the mobile charges and their residues correspond to the charge values.

$$\mathrm{Res}(H, \epsilon_\alpha) = \frac{1}{2\pi i} \tag{2.32}$$

In other words, we are looking for a function that for a finite number of rapidities looks like:

$$H(w) = \prod_j \frac{2}{w - w_j} = \frac{1}{g} + \sum_\alpha \frac{1}{w - \epsilon_\alpha} \tag{2.33}$$

where the w_js are the positions of the roots in the ground state. In the thermodynamic limit, all sources come closer and closer one to the other and a line of discontinuity Γ is created. The value of the field on the two sides of the cut provides the charge density of this region, or:

$$r(w)|dw| = \frac{1}{2\pi i}\left(H(w^+) - H(w^-)\right) dw \tag{2.34}$$

where we denoted $w^\pm = w \pm \epsilon$ for some vanishingly small ϵ. In order to achieve this, we should look for a double-valued function with a cut along the curve. A good candidate is:

$$E(w) = \sqrt{(w - \mu - i\,\Delta_{\rm BCS})(w - \mu + i\,\Delta_{\rm BCS})} \tag{2.35}$$

Moreover, since the position of mobile charges is fixed by the Richardson equations (1.119), from (2.33) we can argue that the final form of the field will be:

$$H(w) = \sqrt{(w-\mu-i\,\Delta_{\rm BCS})(w-\mu+i\,\Delta_{\rm BCS})}\int_\Omega \frac{\varphi(\epsilon)}{\epsilon-w}d\,\epsilon \tag{2.36}$$

where φ is an analytic function that is required to possess all the moments:

$$\int_\Omega \epsilon^m \varphi(\epsilon)d\epsilon < \infty \qquad m \in \mathbf{N} \tag{2.37}$$

To determine this function, we make use of (2.31) and define a closed contour L that encloses all the "pairons", but not the "orbitons" positions. It follows that:

$$\begin{aligned} P\int_\Gamma \frac{r(v)}{w-v}|dv| &= \int \frac{dv}{2\pi i}\frac{\sqrt{(v-\mu-i\,\Delta_{\rm BCS})(v-\mu+i\,\Delta_{\rm BCS})}}{w-v}\int_\Omega \frac{\varphi(\epsilon)}{\epsilon-v}d\,\epsilon \\ &= \int_\Omega \varphi(\epsilon)d\epsilon - \int_\Omega \frac{\varphi(\epsilon)\sqrt{(\epsilon-\mu-i\,\Delta_{\rm BCS})(\epsilon-\mu+i\,\Delta_{\rm BCS})}}{\epsilon-w} \end{aligned} \tag{2.38}$$

in which the first integral contains the poles at infinity and the second the ones in Ω. Plugging (2.38) into (2.31), we find that the unknown function φ is determined to be

$$\varphi(\epsilon) = \frac{\rho(\epsilon)}{\sqrt{(\epsilon-\mu-i\,\Delta_{\rm BCS})(\epsilon-\mu+i\,\Delta_{\rm BCS})}} \tag{2.39}$$

$$\frac{1}{2G} = \int_\Omega \frac{\rho(\epsilon)}{\sqrt{(\epsilon-\mu-i\,\Delta_{\rm BCS})(\epsilon-\mu+i\,\Delta_{\rm BCS})}} = \int_\Omega \frac{\rho(\epsilon)}{\sqrt{(\epsilon-\mu)^2+{\Delta_{\rm BCS}}^2}} \tag{2.40}$$

therefore

$$H(w) = \sqrt{(w-\mu-i\,\Delta_{\rm BCS})(w-\mu+i\,\Delta_{\rm BCS})}\int_\Omega dh \frac{\rho(\epsilon)}{(\epsilon-w)\sqrt{(\epsilon-\mu-i\,\Delta_{\rm BCS})(\epsilon-\mu+i\,\Delta_{\rm BCS})}} \tag{2.41}$$

whose value at infinity is fixed by (2.40). We easily recognize in the latter the BCS gap equation (2.21) in the continuum limit. We now manipulate (2.29) as:

$$\begin{aligned} 2M &= \int_L \frac{dw}{2\pi i}\epsilon(w) = \int_\Omega \frac{\rho(\epsilon)}{\sqrt{(\epsilon-\mu)^2+{\Delta_{\rm BCS}}^2}}\int_L \frac{dw}{2\pi i}\frac{\sqrt{(w-\mu)^2+{\Delta_{\rm BCS}}^2}}{\epsilon-w} \\ &\int_\Omega \frac{\rho(\epsilon)}{\sqrt{(\epsilon-\mu)^2+{\Delta_{\rm BCS}}^2}}\left(\sqrt{(\epsilon-\mu)^2+{\Delta_{\rm BCS}}^2}-(\epsilon-\mu)\right) \\ &= \int_\Omega \rho(\epsilon)\left(1-\frac{(\epsilon-\mu)}{\sqrt{(\epsilon-\mu)^2+{\Delta_{\rm BCS}}^2}}\right) \end{aligned} \tag{2.42}$$

in which the second term comes from the residue $(\epsilon-\mu)$ of the second-order pole of the function $E(w)/w$ at infinity, after deforming the contour of integration in order to encircle the interval Ω. This equation corresponds to the number equation (2.23) in the BCS theory. Following an analogous path for (2.30), we obtain that the ground state energy is:

$$E_0 = \int \frac{dw}{2\pi i}wH(w) = \int_\Omega \epsilon\rho(\epsilon)\left(1-\frac{\epsilon-\mu}{\sqrt{(\epsilon-\mu)^2+{\Delta_{\rm BCS}}^2}}\right)dh - \frac{{\Delta_{\rm BCS}}^2}{4G} \tag{2.43}$$

One has, for instance, that at vanishing interaction between pairs, the pair chemical potential (twice as much as the single electron chemical potential) is just equal to the Fermi energy $\epsilon_F = 2\mu$, i.e., the system is composed of noninteracting fermions doubly occupying the lowest energy shells.

What can the Richardson model tell us for a nanoscopic superconductor? When is g already big enough for the system to show superconducting behaviour? For answering to these question, we need the analysis of [28, 45], which is based on a $1/M$ expansion and on the electrostatic analogy of section 1.4.3. This generalizes the results above and shows that the first order in $1/M$ yields the discrete form of the BCS equations (2.21,2.23).

One way of characterizing the superconducting behaviour is the presence of a gap in the spectrum, which can be computed from the Bethe roots (see eq. 2.26). Here and in the following, in order to account for a finite single-particle level spacing, we need to define an intensive Richardson gap [36], which is related to the BCS gap by

$$\Delta_{\mathrm{BCS}} = N\Delta \tag{2.44}$$

Note that the LHS is the parameter which can be extracted from the root configuration. The gap is roughly proportional to $N\,g$. If one chooses as a criterion the condition $\Delta_{\mathrm{BCS}} \simeq d$, then this is met for $g^* \sim 1/N$. It is also possible to exploit the analysis of [28, 45], which is based on the electrostatic analogy the Richardson equations [44, 29]. In the large–N limit, the parameters of the model satisfy the BCS equations:

$$N - 2M = \sum_\alpha \left(1 - \frac{\varepsilon_\alpha - \mu}{\sqrt{(\varepsilon_\alpha - \mu)^2 + \Delta_{BCS}^2}} \right) \tag{2.45}$$

and

$$\frac{1}{2g} = \sum_\alpha \frac{1}{\sqrt{(\varepsilon_\alpha - \mu)^2 + \Delta_{BCS}^2}} \tag{2.46}$$

A more refined way is to use the fact that the gap is directly related to the occupation number of each level since, as we saw, the superconducting ground state is characterized by a smoothing the Fermi surface arising from the scattering of the pairs. Then, following [46], one can consider the order parameter

$$\Psi = 2\sum_\alpha u_\alpha v_\alpha \tag{2.47}$$

which reaches its saturation value (unit value) when the occupation of the levels is uniform over all the energies. This would actually be a condition for strong superconductivity. Operatively, one can obtain the Bogolubov parameters in the expression above from the expectation values:

$$\left\langle b_\alpha^\dagger b_\alpha \right\rangle = v_\alpha^2 \,, \quad \left\langle b_\alpha b_\alpha^\dagger \right\rangle = u_\alpha^2 \,, \quad \left\langle b_\alpha^\dagger b_\beta \right\rangle = u_\alpha v_\alpha u_\beta v_\beta \tag{2.48}$$

and also:

$$u_\alpha v_\alpha = \sqrt{\left\langle S_\alpha^- S_\alpha^+ \right\rangle \left\langle S_\alpha^+ S_\alpha^- \right\rangle} = \sqrt{1/4 - \left\langle S_\alpha^z \right\rangle^2} \tag{2.49}$$

Computation of the right-hand side by the aid of (1.128) allows to estimate (2.47). This is plotted in figure 2.1 at half filling: assuming as a threshold a value of $\Psi^* = 1/2$, we can claim that the system shows strong superconducting behaviour for $g > g^* \simeq 0.25$. Finite-size effects play a negligible role, in this case, as argued in [36]. The superconducting parameter is related to the gap as

$$\Psi = 2\frac{\Delta_{\mathrm{BCS}}}{g} = 2\frac{N\Delta}{g} \tag{2.50}$$

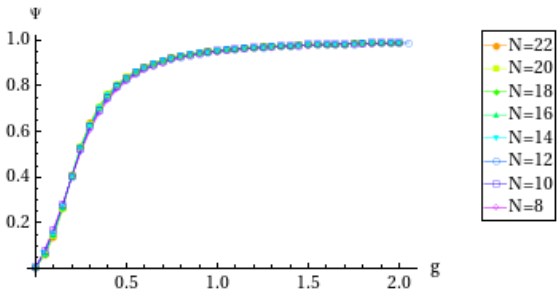


Figure 2.1: Order parameter at different sizes at half filling

2.2 Tunnelling currents in fermionic superfluids and bosonic condensates

A striking feature of superconducting metals is that when connected with a low-resistance junction, a current can flow among them even in the absence of an applied voltage bias. This phenomenon is called a Josephson current [47, 48]. Examples of such settings are the so-called "SNS" junctions: two superconductors are separated by a thin metal in the normal state. Due to the diffusion of Cooper pairs into the metallic layer, the inset becomes weakly superconducting, which realizes a Josephson junction. Other possible settings are the Superconductor-Quantum Dot-Superconductor (S-QuDot-S) Josephson junctions, which are implemented by the use of carbon nanotubes, but many more examples are known.

2.2.1 The Josephson current

The passage of electrons from one lead to the other is the result of the penetration of the electron wavefunction through the junction, therefore a consistent theory should deal with the system as a whole. In facts, formation of Cooper pairs of electrons belonging to different metals is possible. This leads to the possibility of pair tunnelling with a probability which is comparable with that of a single electron and on the appearance of a condensate current, that can flow across the junction even in the absence of applied voltage.

A microscopic derivation (see [40]) can be given with the method of the tunnelling Hamiltonian. We shall consider two superconducting grains coupled by a weak Josephson tunnelling term and study the Josephson current between the two. The Hamiltonian is written as:

$$\mathcal{H} = H_L + H_R + \lambda\, \Delta_{\mathrm{BCS}}\, H_T \tag{2.51}$$

where H_L and H_R are two Richardson Hamiltonians (1.86), Δ_{BCS} is the BCS gap and H_T is a fermion tunnelling term

$$H_T = -\sum_{\sigma=\uparrow,\downarrow}\sum_{\alpha,\beta}\left(T_{\alpha,\beta}c^{\dagger}_{\alpha\sigma,L}c_{\beta\sigma,R} + h.c.\right) \tag{2.52}$$

which conserves the total number of electrons within the two-lead system, but energetically favours the states which hybridize different fermion numbers on the two sides. The total

current from the left to the right is equal to the rate of decrease of the number of electrons in the left metal, multiplied by the electron charge. Then

$$J = -e\dot{N}_L = -\frac{ie}{\hbar}\left[H, N_L\right] = -\frac{ie}{\hbar}\left[H_T, N_L\right] \tag{2.53}$$

where the last equality holds because the tunnelling term is the one that leads non conservation of the number of electrons in the left and right lead separately. Substituting its explicit form, one has

$$J = \frac{ie}{\hbar}\sum_{\sigma=\uparrow,\downarrow}\sum_{\alpha,\beta}\left(T_{\alpha,\beta}c^{\dagger}_{\alpha\sigma,L}c_{\beta\sigma,R} - h.c.\right) \tag{2.54}$$

We now suppose a small value of λ, much smaller than the other two scales involved, i.e., d and g.

Perturbation theory on the ground state of the system, yields

$$|\Psi_0\rangle = |\Phi_0\rangle + \sum_m \frac{(H_T)_{m,0}}{E_0^{(0)} - E_m^{(0)} + i0}\,|\Psi_m\rangle \tag{2.55}$$

where the sum runs over the excites states of the system and the superscript 0 identifies the factorized ground state of the two separate Hamiltonians. It follows that

$$\langle J\rangle = \sum_m \frac{(H_T)_{m,0}}{E_0^{(0)} - E_m^{(0)} + i0}\,\langle\Psi_0|J|\Psi_m\rangle + c.c. \tag{2.56}$$

The tunnelling term is written by creation operators for single electrons. On the other hand, we have seen that a convenient description for the ground end the excited states is in terms of quasiparticles. It follows that there are two families of intermediate states which are involved in the process. The first kind of states are those which transfer one quasiparticle back and forth and that conserve the total number of quasiparticles. These states give rise to a *normal* current, which vanishes if there is no applied voltage. The second kind of intermediate states contains one more or one less quasiparticle on both sides of the system, even if the total number of electrons is unchanged. These processes give rise to a voltage-independent Josephson current.

and using the explicit expression for the current operator (2.53), one sees that the terms of the first family are those that contain the product $T^2_{\alpha,\beta}$. One is left with the differences

$$\frac{1}{E_0^{(0)} - E_m^{(0)} + i0} - \frac{1}{E_0^{(0)} - E_m^{(0)} - i0} = -2\pi i\delta(E_0^{(0)} - E_m^{(0)}) \tag{2.57}$$

And the final result is

$$\begin{aligned}\langle J\rangle \;=\; & \frac{2\pi e}{\hbar}\sum_{m,\sigma,\alpha,\beta}|T_{\alpha,\beta}|^2\delta(E_0^{(0)} - E_m^{(0)}) \\ & \left[\left(c_{\alpha,L}c^{\dagger}_{\beta,R}\right)_{0,m}\left(c^{\dagger}_{\alpha,L}c_{\beta,R}\right)_{m,0} - \left(c^{\dagger}_{\alpha,L}c_{\beta,R}\right)_{0,m}\left(c_{\alpha,L}c^{\dagger}_{\beta,R}\right)_{m,0}\right]\end{aligned} \tag{2.58}$$

where we recall that indexes α, β label single-particle levels, σ is for spin, while the latin index m is for a full many-body state of the unperturbed Hamiltonian, while the subscripts of the round brackets refer to the states upon which the matrix element is computed. Reminding

that $n_{\mathbf{k}}$ denotes electronic occupation number in the ground state, the expression in brackets results in

$$n_{\alpha,L}(1-n_{\beta,R})-(1-n_{\alpha,L})n_{\beta,R}=n_{\alpha,L}-n_{\beta,R}$$

If the two occupation distributions are the same, then summation in (2.58) provides a null result. The only way to have a normal current is then evidently to shift the relative energies on the two sides by applying a voltage difference to the junction. Then the current becomes

$$\langle J\rangle \;=\; \frac{2\pi e}{\hbar}\sum_{\sigma,\alpha,\beta}|T_{\alpha,\beta}|^2\left[n(\epsilon_\alpha - eV)-n(\epsilon_\beta)\right] \tag{2.59}$$

Let us go back to (2.54) and consider the other contribution. The expression for the current expectation value also contains terms of the form

$$i\frac{e}{\hbar}\sum_{\alpha,\beta,\sigma,m}\Big[T_{\alpha\,\beta}T_{\gamma\,\delta}\left(c^\dagger_{\alpha,\sigma,L}c_{\beta,\sigma,R}\right)_{0,m}\left(c^\dagger_{\gamma,\sigma',L}c_{\delta,\sigma',R}\right)_{m,0}\left(\frac{1}{E_0^{(0)}-E_m^{(0)}+i0}+\frac{1}{E_0^{(0)}-E_m^{(0)}-i0}\right)$$
$$+T^*_{\alpha\,\beta}T^*_{\gamma\,\delta}\left(c_{\alpha,\sigma,L}c_{\beta,\sigma,R}\right)_{0,m}\left(c^\dagger_{\gamma,\sigma',L}c^\dagger_{\delta,\sigma',R}\right)_{m,0}\left(\frac{1}{E_0^{(0)}-E_m^{(0)}+i0}+\frac{1}{E_0^{(0)}-E_m^{(0)}-i0}\right)\Big] \tag{2.60}$$

We would like to diagonalize the quadratic expression above by the use of a Bogolubov transformation [49]. However, in contrast to the thermodynamic quantities, the Josephson current depends on the phases of the condensates on the two leads. Therefore, the coefficients u, v are allowed to be complex in:

$$c_{\beta,+}=u_\beta A_\beta+v^*_\beta B^\dagger_\beta\,,\qquad c_{\beta,-}=u_\beta B_\beta-v^*_\beta A^\dagger_\beta \tag{2.61}$$

with the constraint $|u_\beta|^2+|v_\beta|^2=1$ in order for the new operators to satisfy ordinary anticommutation rules. The next steps, for each of the two grains, can be carried on as above and yield:

$$|v_\beta|^2=\frac{1}{2}\left(1-\frac{\epsilon_\beta}{\sqrt{\epsilon_\beta^2+|\,\Delta_{\rm BCS}\,|^2}}\right)\,,\qquad |u_\beta|^2=\frac{1}{2}\left(1+\frac{\epsilon_\beta}{\sqrt{\epsilon_\beta^2+|\,\Delta_{\rm BCS}\,|^2}}\right) \tag{2.62}$$

with the (complex) BCS gap function

$$\Delta_{\rm BCS}=g\sum_\alpha u_\alpha v^*_\alpha \tag{2.63}$$

The phase of the Bogolubov coefficients can be explicitly factored out

$$u_\beta=|u_\beta|e^{i\phi}\,,\qquad v_\beta=|v_\beta|e^{i\phi} \tag{2.64}$$

To proceed, we would like to compute the average of the operator at given temperature T, so that one needs to average over many states, weighted with the Gibbs distribution factor. However, one can use the fact that in realistic systems of weakly interacting fermions, composed of a large number of particles, the occupation number of states within a given energy window is the same. Up to small fluctuations, this is the Fermi distribution $n_F(\epsilon)$ at given temperature.

We then make the hypothesis that $T_{\alpha,\beta}$ is real. When there is no applied voltage, the expectation value of the current is found to be

$$\begin{aligned}\langle J\rangle &= \frac{e\,g}{\hbar}\sum_{\mathbf{k},\mathbf{q}}|u_{\mathbf{k}}v_{\mathbf{k}}u_{\mathbf{q}}v_{\mathbf{q}}|T^2_{\alpha,\beta}\sin(\phi_L-\phi_R)\\ &\quad\left[\frac{n_{\mathbf{k}}(1-n_{\mathbf{q}})}{\epsilon_{\mathbf{k}}-\epsilon_{\mathbf{q}}}-\frac{n_{\mathbf{k}}n_{\mathbf{q}}}{\epsilon_{\mathbf{k}}+\epsilon_{\mathbf{q}}}+\frac{(1-n_{\mathbf{k}})(1-n_{\mathbf{q}})}{\epsilon_{\mathbf{k}}+\epsilon_{\mathbf{q}}}-\frac{(1-n_{\mathbf{k}})n_{\mathbf{q}}}{\epsilon_{\mathbf{k}}+\epsilon_{\mathbf{q}}}\right]\end{aligned} \tag{2.65}$$

This is already an exact expression that shows the *Josephson relation*. If the two superconducting leads show phase coherent behaviour, the overall phase difference is well-defined. The Josephson current only depends on it through the law

$$J = J_c\sin(\phi_L-\phi_R) \tag{2.66}$$

which is expected to hold at all times. The *critical current* j_c is model dependent. For instance, it can be computed to be

$$J_c = \frac{\pi\,\Delta_{\mathrm{BCS}}}{2eR} \tag{2.67}$$

for two equal infinite leads at temperature $T=0$, connected by a conductor with resistance R [48, 40].

2.3 The BEC-BCS crossover

In conventional superconductors (like Al, Hg, Sn, ...), electrons with opposite spin can form *Cooper pairs* at temperatures below the superconducting critical temperature under the condition that an attractive interaction among them exists. In this superconductors, the average "size" of the pair ξ_{pair} exceeds by some orders of magnitude the typical interparticle distance k_F^{-1} and turns out to be about $\xi_{pair}k_F\sim 10^3-10^5$. This means that the different pair are highly overlapping with all the others. The discovery of high-temperature cuprate superconductors has changed completely the picture of Cooper pair superconductivity, as in these materials the size of the pair is about $\xi_{pair}k_F\sim 5-10$. In other words, the description of the pairs lies somewhere between strongly attracting, tightly bound pairs of fermion forming composite bosons and overlapping, loosely correlated Cooper pairs (see [50]).

Therefore, the investigation of the crossover between these two situation (BCS-BEC crossover) has become of great interest to the condensed matter community, and has received further impulse by the development, on the experimental side, of the techniques and the machineries to investigate ultracold trapped Fermi atoms. In particular, it is possible to achieve in the laboratory the tuning of the effective attractive interaction between fermions of different species, therefore realizing the crossover. This is achieved by the use of the Fano-Feshback resonances, which are characterized by a resonant coupling between a two-atom scattering state with vanishing energy and a bound state in a closed channel.

A simple description of the BCS-BEC crossover in an homogeneous system can be given at zero temperature in the approximation (2.24). The low-energy physics is encoded in the s-wave scattering length a, which is related to the bare fermion coupling strength (see [42], sec. 35) by:

$$g = \frac{4\pi\hbar^2 a}{m} \tag{2.68}$$

for the interaction (2.24). It is given by the integral:

$$-\frac{mgV}{4\pi a}+1=-\frac{gV}{2}\int\frac{d^3k}{(2\pi)^3}\frac{1}{\epsilon_k} \tag{2.69}$$

which is divergent whenever a cutoff (i.e., a Debye frequency) is not present. Nevertheless, combining it with (2.24) and the integral form of (2.46), one obtains a convergent expression in terms of the scattering length itself:

$$\frac{mg}{4\pi a}=-\frac{g}{2(2\pi)^3}\int d^3\mathbf{k}\left(\frac{1}{\sqrt{\xi_{\mathbf{k}}^2+{\Delta_{\rm BCS}}^2}}-\frac{1}{\epsilon_k}\right) \tag{2.70}$$

This equation, together with (2.45), can be used to characterize the crossover. In the BCS limit $a\to 0^-$, and the solution can be found to be $\mu\simeq\epsilon_F>0$ and $\Delta_{\rm BCS}\propto\epsilon_F\,e^{-\pi/2k_Fa}$. Instead the BEC limit is characterized by $\mu\leq 0$, $a\to 0^+$ and one finds

$$n\simeq\frac{\Delta^2(2m)^{3/2}}{16\pi\sqrt{|\mu|}}\,,\qquad\frac{m}{4\pi a}\simeq\frac{(2m)^{3/2}\sqrt{|\mu|}}{8\pi}\left(1+\frac{\Delta^2}{16\mu^2}\right) \tag{2.71}$$

which imply for the chemical potential

$$\mu\simeq-\frac{1}{2ma^2}+\frac{a\pi}{m}n \tag{2.72}$$

If we accept the picture of a pair of fermions binding together to form a boson, then we can set [51, 52] the number of bosons to be $M=n/2$, their mass $m_B=2m$ and the scattering length $a_B=2a$. The chemical potential for the pairs is $\mu_B=2\mu+\epsilon_0$, with the definition of the molecular binding energy $\epsilon_0=1/(ma^2)$. Then (2.72) can be seen as the equation of state

$$M=\frac{m_B}{4\pi a_B}\mu_B\simeq\frac{\mu_B}{g_B} \tag{2.73}$$

which is associated to a system of superfluid bosons and can be derived from the Gross-Pitaevski formalism [51, 52] and the last relation comes from (2.68). In other words, varying the scattering length through the bare coupling induces a crossover between the physics of fermions in a superconducting state to a bosonic condensate.

In the case of attracting fermions, one has $\mu>0$, $a<0$ and the physics described by the system is the one associated with the BCS theory. The vanishing of the left-hand side corresponds to a diverging scattering length and is called *unitary* limit [53]. It is associated to a change of sign in the scattering length, which in turns signals the formation of metastable pairs of fermions and, for growing attraction, of stable bosonic pairs [51, 50].

Previous work about the tunnelling current through the BEC-BCS crossover [54], involving the numerical solution of the Bogolubov-De Gennes equations, has shown two important aspects of the crossover. First, that the Josephson relation (2.66) is modified when passing from a fermionic superfluid to a true Bose condensate. Second, that the maximum current attainable for a given barrier shape between the two sides of the system is maximum around (not exactly at) unitarity.

The BCS-BEC crossover can be argued in the framework of the integrable Richardson model. The issue was first tackled in [55], where the model (1.88) was considered in the

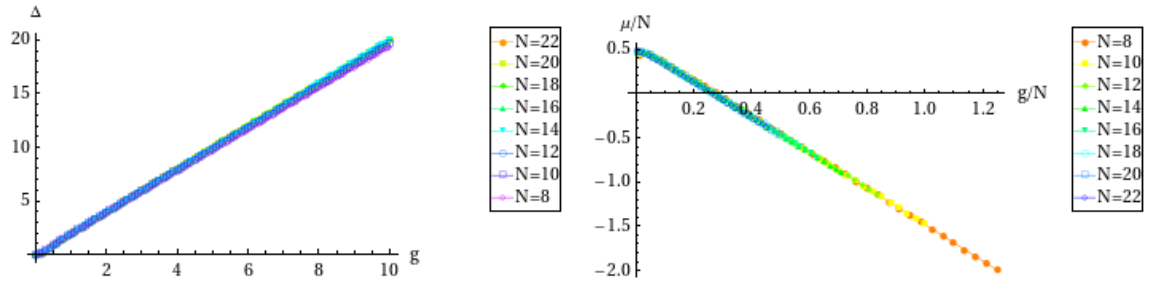


Figure 2.2: Dependence of the intensive gap and chemical potential per particle from the bare pairing strength g, as computed from the ground state root configuration for $M = N/2 - 1$ and the discrete version of the BCS equations in systems with different sizes. The chemical potential for the sizes $N = 14 - 22$ on the rightmost part of the graphic is computed by extrapolating the intensive gap.

thermodynamic limit and it was suggested there that root configurations at strong enough coupling can be used to identify the boundaries of the crossover.

A generic feature of the crossover from a fermionic to a bosonic behaviour of the constituents of the gas is that the chemical potential μ must change sign. In our canonical model, the number of particles M is kept fixed and μ is not a free parameter, but is fixed from (2.45), which we can solve for the chemical potential. Note that the correct setting for a few–body problem would be that of solving for a time–dependent chemical potential. However, having in mind application to large system, we keep the average expectation value, the procedure standing as an approximation for which the error scales as the inverse of the size.

Summing up, given a root configuration computed "evolving" the lowest M energy levels to a given value of g, the gap and the chemical potential are determined. The results are shown in Figure 2.2. Note that, whenever $M < N/2$, the chemical potential becomes more and more negative while increasing g: at some point, it crosses the real axis to negative values, suggesting the presence of a crossover. In the BCS scaling, in which the level spacing goes to zero as the inverse of the size, the crossing point tends to $g* = 0.5$ in the thermodynamic limit, whereas in our equally–spaced model, we have $g^* \propto N$. In all cases, we remind that the single–particle levels can be chosen arbitrarily and in particular that they can be translated: the value for which the chemical potential crosses the real axis is only conventionally identified as an indication of the crossover. As a matter of fact, what we learn from the model is that the chemical potential varies from the Fermi energy when $g \to 0$ to arbitrarily negative values as the bare attraction between fermions is increased.

It is also possible to have some insight on the crossover from the functional form of the eigenstates themselves, if one goes back to the explicit one-particle states (see 1.113,1.117). The Richardson model has no notion of space, in its bare formulation. Nevertheless, having in mind the description of a lead, it is clear that the energy levels must be associated with the momentum bands of the conduction electrons through a quadratic dispersion relation $\varepsilon_k = \hbar^2 k^2/(2m)$. Analogously, the spatial variation of the wavefunction can be encoded in the phase factor $e^{i\mathbf{k}\cdot\mathbf{r}}$.

According to (1.121), the energy of a state is simply given by the sum of the Richardson roots. It is then tempting to interpret such roots (or their real part) as squared momenta. It

follows that the single-pair wavefunction can be written as:

$$\varphi_w(\mathbf{r}) \sim \sum_k \frac{e^{i\sqrt{2\,m\,w}r}}{w - \varepsilon_k} \tag{2.74}$$

with m the effective mass of the pair. Then, in the noninteracting limit $\sqrt{2\,m\,w} \simeq k_f$, so that the wavefunction is delocalized over the whole sample; in the strong-coupling limit, instead, the ground state has $w \simeq -\frac{1}{\xi^2}g$ with some positive "localization length" ξ and the wavefunction is exponentially localized.

Tunnelling on the BEC side

The macroscopic condensate wavefunction $\Psi = \Psi(r,t)$ of a Bose-Einstein condensate obeys a nonlinear equation known as Gross–Pitaevski equation.

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + g\,|\Psi|^2\,\Psi \tag{2.75}$$

with g a bare interaction strength. A fairly general description of the tunnelling of bosons between trapped condensates can be given by considering a two–state model in which a uniform amplitude over each sample is taken, modulated by a phase [56]. A variational ansatz for a system of two weakly linked condensates is

$$\Psi(r,t) = \psi_1(t)\Phi_1(r) + \psi_2(t)\Phi_2(r) \tag{2.76}$$

Given that $M_{1,2}$ particles are present in the two sides of the sample, with phases $\phi_{1,2}$ one can chose a model amplitude like $\psi_{1,2} = \sqrt{M_{1,2}}e^{i\phi_{1,2}}$. The resulting equations are:

$$i\hbar\frac{\partial\psi_1}{\partial t} = \left(E_1^0 + U_1M_1\right)\psi_1 - K\psi_2 \tag{2.77}$$

$$i\hbar\frac{\partial\psi_2}{\partial t} = \left(E_2^0 + U_2M_2\right)\psi_2 - K\psi_1 \tag{2.78}$$

where K is the coupling matrix element. The parameters $E_{1,2}$, $U_{1,2}$ can be derived by substituting the ansatz (2.76) into (2.75) and are provided in the original papers [56, 57], but we will not need their expression in the following. By substituting the ψ and massaging the system, one arrives at the couple of differential equations:

$$\frac{\partial z}{\partial 2Kt} = -\sqrt{1-z^2}\sin\phi \tag{2.79}$$

$$\frac{\partial\phi}{\partial 2Kt} = \Lambda z + \frac{z}{\sqrt{1-z^2}}\cos\phi + \delta E \tag{2.80}$$

for the fractional occupation difference $z = \frac{M_1-M_2}{M_1+M_2}$ and the relative phase $\delta\phi = \phi_1 - \phi_2$. The dimensionless parameters that appear are

$$\delta E = \frac{E_1^0 - E_2^0}{2K} + \frac{U_1 - U_2}{4K}M_T \tag{2.81}$$

$$\Lambda = \frac{U_1 + U_2}{4K}M_T \tag{2.82}$$

and $M_T = M_1 + M_2$ is the total number of bosons.

An useful mechanical analogy to be noticed is that the system (2.79) can be seen as derived from the Hamiltonian of a non rigid pendulum

$$H = \frac{\Lambda}{2} z^2 - \sqrt{1 - z^2} \cos\phi + \delta E z \tag{2.83}$$

in which the time evolution of the conjugated variables ϕ, z can be found from:

$$\dot{z} = -\frac{\partial H}{\partial \phi} \qquad \dot{\phi} = \frac{\partial H}{\partial z} \tag{2.84}$$

We can rephrase these relations by defining a fake "angle" variable θ, such that $z = \sin\theta \in [-1, 1]$. It follows that

$$\dot{\theta} = -\sin\phi \tag{2.85}$$

$$\dot{\phi} = \tan\theta \cos\phi + \delta E + \Lambda \sin\theta \tag{2.86}$$

this form explicitly shows that the instantaneous relative occupation is a function of time only through the relative phase ϕ.

Note that a simple form is obtained as long as the number of particles in the system tends to infinity, being $\Lambda \propto M_T^{-1}$, whatever the ratio of the interactions over the tunnelling parameters may be.

Starting from (2.85) and identifying with a prime the derivative with respect to ϕ, we write

$$\frac{d\theta}{dt} = \frac{d\theta}{d\phi}\frac{d\phi}{dt} = \theta' \tan\theta \cos\phi = -\sin\phi$$

which implies

$$\tan\theta \frac{d\theta}{d\phi} = -\tan\phi$$

This is easily integrated, yielding the dependence

$$\cos\theta = \frac{A_0}{\cos\phi} \tag{2.87}$$

where the constant $A_0 = \cos\phi_0 \cos\theta_0$ is fixed by the initial conditions. Substituting into (2.79) one has that the dependence of the current on the phase is simply:

$$I(\phi) = \frac{M}{2}\dot{z} = \frac{M}{2}\cos\theta\dot{\theta} = -\frac{M\,A_0}{2}\tan\phi \tag{2.88}$$

We will see that this picture is partially recovered in our numerical analysis. However, for our small particle unbalances, we have to remark that the variable z appears not to be the only relevant quantity in the dynamic.

2.4 Study of coupled Richardson models from integrability

We show in the following that, remarkably enough, integrability can be crucial in simplifying the problem even when the model under study is non-integrable. In particular, we will couple two Richardson Hamiltonian through a fermionic tunnelling term and study the phase relation and the current. The interest lies both in the fact that many observed properties are expected to hold also in the thermodynamic limit, therefore for real superconducting leads, and in the application in cold-atoms experiments, where few-body Hamiltonians can be explicitly investigated.

2.4.1 The Hamiltonian

We are interested in studying the Hamiltonian (2.51), in the approximation in which all matrix elements are equal. The tunnelling term is then

$$H_T = -\sum_{\sigma=\uparrow,\downarrow}\sum_{\alpha,\beta}\left(c^{\dagger}_{\alpha\sigma,L}c_{\beta\sigma,R} + h.c.\right) \tag{2.89}$$

The present work shows the exact numerical dynamics of the system. With slight abuse of terminology, we shall refer to each side of the system as "grain"[1].

We will now argue, following ([58]) that for small values of λ, an effective Hamiltonian can be written only in terms of the pair operators, therefore greatly simplifying the problem. Since the single-site Hamiltonian contains only interactions among pairs, eigenstates of (1.86) are classified in terms of their seniority ν, i.e., the number of the unpaired electrons. In the regime $(\lambda\,\Delta_{\mathrm{BCS}}/d)^3 \ll 1$, the second order effective tunnelling term can be written as:

$$H_2 = -\sum_{\sigma}\sum_{\nu}\sum_{\alpha\beta} H_T \frac{|\alpha_L\beta_R\sigma;\nu\rangle\langle\alpha_L\beta_R\sigma;\nu|}{E_{\alpha_L\beta_R\nu}} H_T \tag{2.90}$$

in which the sum runs over all the possible intermediate states that can be reached from a ν-seniority couple of states $|N/2+\nu\rangle_L \otimes |N/2+\nu\rangle_R$, by removing an electron of spin σ from the level β_R of the right grain and adding it on the level α_L on the left grain (or viceversa). The quantity $E_{\alpha_L\beta_R\nu}$ is the corresponding excitation energy relative to the initial state.

We will now try to limit the space of states the intermediate sum runs over to the lowest energy ones, having in mind to act with (2.90) on the lowest-energy states of the two grains, in which all electrons are bound into Cooper pairs.

The energy $E_{\alpha_L\beta_R\nu}$ will include the energy necessary for moving an electron from the starting level to an intermediate level on the other side, the energy needed to break a pair and the effect of the blocking of the states on the collective excitations on both sides[2]. In other words, it is the energy of a *collective excitation*, arising from the fact that the levels α and β are singly occupied. This limits the space of intermediate states to the ones with the lowest energy, in which such collective excitations are quasiparticles (2.5), created on top of the superconducting vacuum.

In the BCS regime we expect that the breaking of a pair associated with the tunnelling of a single electron to be energetically costly. When going to the second order, it is more convenient to the system to regain the gap energy by the tunnelling of a second electron among the two levels, in order to reconstruct the Cooper pair in the other grain [47]. On the other hand, we have seen that single-electron tunnelling does not produce a current in the absence of an applied driving force. To second order, in (2.90) two kinds of processes can happen, which are summarized in Figures 2.3,2.4.

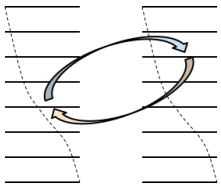


Figure 2.3:

In the first kind of processes, one electron on one side undergoes a transition to a level on the other side, then is re-created on the

[1] For more realistic modelling of nanograins, their high charging energy should be taken into account

[2] In principle, the charging energy due to the transfer of one unit charge to the grain should be taken into account, due to the limited number of levels in our system. We address to the situations in which one can neglect such charging energy, or to experimental settings in which ultracold *neutral* atoms can tunnel among two neighbouring traps

same level of the starting grain. On the other hand, *coherent* pair tunnelling involves both electrons of a pair and can be written in term of the *bosonic* operators $b^\dagger_{\alpha,L} b_{\beta,R}$ or $b_{\alpha,L} b^\dagger_{\beta,R}$ directly. Processes in which one electron hops on the other grain and then back on a different level from the starting one are energetically suppressed, since they involve both the breaking of a pair with an energy cost equal to the BCS gap $\Delta_{\rm BCS}$ and the blocking of a level, which affects all the levels above it and has therefore an energy cost roughly proportional to N.

Assuming the two superconductors to have a well-defined phase (which will be checked in the following), the coherent tunnelling involves a phase shift on the state in which it takes place and a corresponding variation of the relative number of particles $\delta M = \pm 2$. Conversely, in the back-and-forth electron tunnelling process the energy shift does not depend on the relative phase, nor any electron or Cooper pair is effectively transported from one side to the other, unless there is an applied bias. The net effect of this last process is then a shift of the single-particle energy levels, due to the second-order coupling to the other grain.

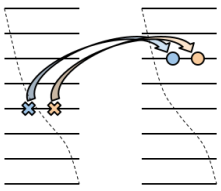


Figure 2.4:

We shall therefore focus on the coherent pair tunnelling, for which the effective Hamiltonian [58] is then written as:

$$H_J = -2(\lambda\,\Delta_{\rm BCS})^2 \sum_{\alpha,\beta} \frac{b^\dagger_{\alpha,L} b_{\beta,R} + b_{\alpha,L} b^\dagger_{\beta,R}}{\sqrt{\xi^2_{\alpha,L} + {\Delta_{\rm BCS}}^2} + \sqrt{\xi^2_{\beta,R} + {\Delta_{\rm BCS}}^2}} \tag{2.91}$$

and the excitation energies have their BCS value with the gap $\Delta_{\rm BCS}$. We remind the reader that the Hamiltonian (1.86) does not involve any single-electron scattering and that the Hilbert space is divided into seniority subspaces. The ground state for a system with an even number of electrons is the one in which all of them are paired and there are no singly-occupied levels. The form (2.91) is then particularly relevant because it formalizes the fact that preparing the system in its ground state and adding a weak *fermionic* tunnelling term to the grain Hamiltonian will not destroy the Cooper pairs picture. This provides an evident simplification in the problem, since, together with (1.88), allows us to study the Josephson junction problem only in terms of hard-core-bosons only, since the subspaces with different seniority will not be accessed neither by the single-site dynamics, nor by the coupling between different sites.

2.4.2 Numerical analysis

The goal of this section is to study the exact dynamics of a two-grain system after switching on of a tunnelling term, extracting the time-dependence of observables such as the Josephson current or the number of pairs in the two grains, as well as the evolution of the bulk phase difference between the superconductors.

The most realistic setting is the one in which the two superconductors are exactly degenerate in energy, and share the same value of the pair scattering strength g. Moreover, we shall chose equally spaced, non-degenerate single-particle energy levels.

Integrability enters the game in that it gives the exact eigenstates of the two uncoupled grains and, most importantly, the exact hopping matrix elements. This is not all, since it also provides an efficient truncation mechanism to select the most important eigenstates in the dynamics.

To see this, we shall consider the two limits $g \to 0$ and $g \to \infty$. In the noninteracting case, the single-level occupation numbers are good quantum numbers for the system. It follows that all the excitations above the Fermi sea ground state induced by the coupling, in the regime in which the Josephson coupling small ($\lambda/d \ll 1$), are the one particle-hole states, obtained from exciting one pair from below to above the Fermi level. In the opposite limit, it is sufficient to consider the spin formulation (1.87) to see that the total spin quantum number of the eigenstate is conserved in the dynamics, since:

$$H_R \to_{g\to\infty} \simeq -g\left(\vec{S}_{tot}\cdot\vec{S}_{tot} - (S^z_{tot})^2 - S^z_{tot}\right) \tag{2.92}$$

and, e.g., at half filling (zero magnetization) has eigenvalues:

$$H_{R,g\to\infty}\,|s,0\rangle \simeq -gs(s+1)\,|s,0\rangle \tag{2.93}$$

In the strong coupling limit, also the effective tunnelling Hamiltonian simplifies, in that the bcs gap diverges linearly with g and all the couples of levels in (2.91) factorize a common term, yielding the simple form:

$$H_{J,g\to\infty} = -\frac{\lambda\,\Delta_{\rm BCS}}{\sqrt{{\Delta_{\rm BCS}}^2+\mu^2}} S^+_{tot,L} S^-_{tot,R} + h.c. \tag{2.94}$$

The ground state is the unique state in which all the rapidities diverge in the strong coupling limit and is the one with highest (total) spin. From the relation (1.136) one can argue that it is sufficient to restrict the single-site Hilbert space to the root configurations with one less (or one more) rapidity and only one more (or one less) rapidity which diverges at large g, i.e., again the ground state of the new sector.

Algorithms for connecting the number of roots that eventually diverge to the initial state configurations have been given in ([36],[59]) and are based on the sizes of contiguous roots or holes.

This class of states is a subset of the one-particle-hole excitations, therefore no other state is needed. We don't have a rigorous argument to proof that this set of states, certainly the most important one, is sufficient to describe the dynamics also in the middle of the crossover. nevertheless, we can compare the results with exact diagonalization (for $N = 6$) or the effect of adding more total spin subspaces to the dynamics (for $N = 8$). In all these tests, the same results were found. A more drastic approximation, like the one adopted in ([60]), seems not to be satisfactory for weak coupling.

2.4.3 Features of the spectrum

The effect of the weak tunnelling on the level spacing depends essentially on the coupling among fermions. In facts one can identify clearly a regime of nearly non-interacting particles, in which the nearly degeneracy of the levels is given by the number of ways of promoting one or more particles in an excited level to obtain a given energy. The latter is a feature of our choice of equally–spaced levels, yet it is the most natural one . In this regime, the perturbation splits the levels of one band as far as the band spacing, hence giving rise to a spectrum in which the original degeneracies are not seen any more.

On the other hand, in the strong coupling regime states group in eigenstates of the total angular momentum (see the spin representation 1.132). Since the distance among the energies of these subspaces is of order g, in this regime, even a tunnelling term of several times the gap cannot mix the different subspaces among them.

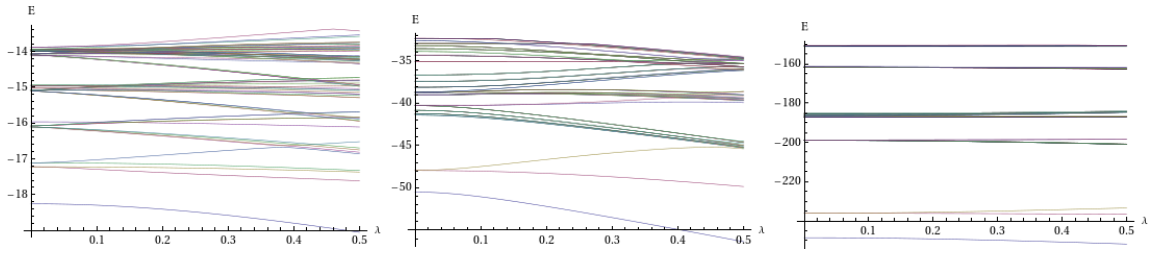


Figure 2.5: Change in the energy levels while varying the tunnelling parameter for a system with $N = 8$ in the half–filled subspace, with $g = 0.1$ (left), $g = 1.2$ (centre), $g = 6.2$ (right).

In the crossover region, the strong coupling subspaces are already quite defined, but not far one from the other. It follows that a sufficiently strong perturbation can still hybridize them.

To make these aspects more quantitative, we may evaluate how much the levels are shifted by turning on λ. This, however, would not convey all the information we have highlighted before: in facts, the absolute value of the shift can be said to be large or small only in relation to another energy scale. This scale is the level spacing in a situation where levels are well-distinguishable (intermediate couplings) and the band spacing in the presence of strong degeneration ($g \to 0$ or $g \to \infty$).

We find therefore more convenient to bin energy levels in classes, in such a way that the band structure is captured. As a second step, we can estimate the change of the distribution when swithching on λ, by considering the difference among the classes for the two values of the perturbation: this tells us whether the original band structure is still intact and whether levels from a band have moved enough that they have come close levels from another band. To be precise, we define

$$\chi_m := (\#\ levels\ in\ the\ m-th\ class)(\lambda) - (\#\ levels\ in\ the\ m-th\ class)(0)$$

Since the number of levels is unchanged, the average of this quantity (with respect to the class index m) is zero. To estimate the change we need therefore to consider its standard deviation σ_χ. The result is that it has a maximum around $g \sim 1$, when the degeneracies of the noninteracting picture are already destroyed, while the energy bands of the strong coupling regime are not evident yet.

For fixed size, the Richardson Hamiltonian describes the physics of a crossover between weakly attracting fermions and strongly coupled bosons. From the point of view of the energy spectrum, this can be seen through the creation of energy bands out of the pair levels, which are more and more separated by increasing g. This is also seen at the level of the coupled spectrum, where the doublet structure characterizing coupled noninteracting systems is melted into an highly–degenerate band structure.

In general, the main difference is for even and odd number of pairs in the system. In the odd case, the ground state is always doubly degenerate without coupling and splits when turning on the perturbation. It is also important to characterize the splitting in their dependence on the gap: it turns out that, as long as the gap opens more and more, the energy difference between the components of the level doublet reaches a maximum splitting.

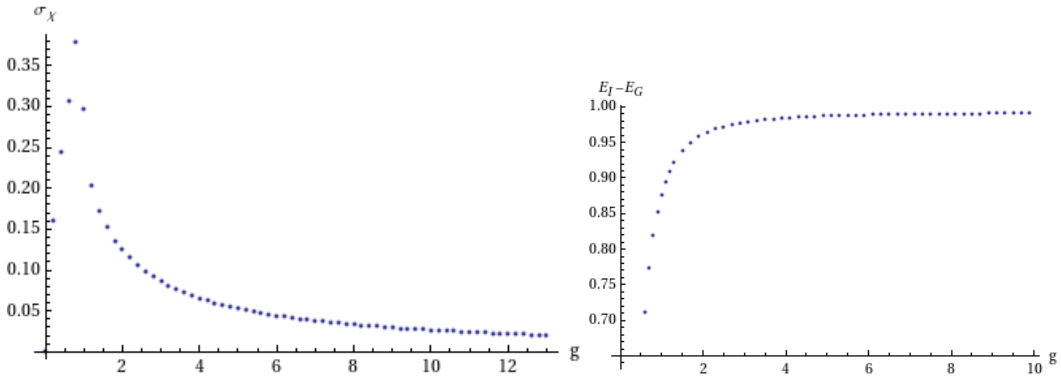


Figure 2.6: Left: estimate of the susceptibility associated with the level structure, as described in the main text, in a system with $N = 8$, $M_T = 7$. Right: energy difference between the first excited state and the ground state, as a function of g at fixed $\lambda = 0.05$.

2.4.4 Definition of the phase relation

Without coupling among grains, an eigenstate of the Richardson Hamiltonian (1.86) can be written as a superposition of eigenstates of the free Hamiltonian (with $g = 0$), where the single levels have a well-defined occupation number. From the expression (1.113) it is clear that the components of any many-particle eigenstates on each level occupation configuration has the same phase at a reference time and at any later time.

An important issue we want to check, however, is whether the grains actually behave as superconducting also when coupled. As a matter of fact, in the presence of a tunnelling term, eigenstates will in principle be written as a combination of many of the factorized states of the two uncoupled Hamiltonian. Nevertheless, as we will see, when the initial particle unbalance is small, the number of involved states is rather small. Moreover, even for higher particle unbalance, when the tunnelling is weak and the pairing strength is strong enough, the Hilbert space of each grain organizes in subspaces, labelled by eigenvalues of the total spin (see section 1.4). It then follows that in most cases, even if the exact states involved are many, the corresponding energy eigenvalues are not very different, therefore the time evolution takes place with nearly definite phase.

Being the overall phase of the single grain undefined, we can only detect phase differences among two configurations in which one particle has been displaced from one level (α) to another (β), by computing the correlation function:

$$\langle \Phi(t)| \, b_\alpha^\dagger b_\beta \, |\Phi(t)\rangle = u_\alpha v_\beta e^{i\phi} \tag{2.95}$$

and being the latter always real, we conclude that the phase difference is always vanishing. Nevertheless, as soon as the two leads are coupled, the initial state will be a complicate superposition of eigenstates, each component evolving with its own eigenenergy

$$|\Psi(t)\rangle = \sum_{\Phi_L, \Phi_R} c_{\Psi, \Phi_L \times \Phi_R}(t) \, |\Phi_L\rangle \, |\Phi_R\rangle \tag{2.96}$$

Therefore, the phase difference ϕ between any two levels in (2.95) will be, in general, a function of time.

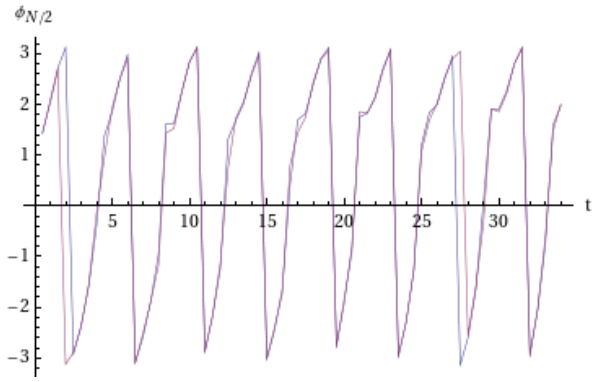


Figure 2.7: Phase difference as determined from correlation functions $z_{N/2}$ and $w_{N/2}$, as defined in (2.98), for two coupled grains with $N = 8$ levels each, total number of pairs $M_T = 8$, pairing strength $g = 0.6$, tunnelling parameter $\lambda = 0.06$ and initial unbalance $z_0 = 0.25$

In our canonical setting, the expectation value $\langle\Psi(t)|\, b_{\alpha,L/R}\,|\Psi(t)\rangle$ is always vanishing, since the operator does not conserve the number of particles. Nevertheless, we can easily recover phase differences between two levels on different leads, by the use of the formalism of Section 1.4. Two-point functions where the operators act on different grains are easily evaluated, while on the same grain can be computed as well [61, 62], but we are going to use the former procedure because it gives a clear quantitative understanding of the amplitude of fluctuation, as described below, not to mention the fact that it is computationally simpler.

From the correlation function

$$\langle\Phi(t)|\, b^\dagger_{\alpha,L} b_{\beta,R}\,|\Phi(t)\rangle = u_{\alpha,L} v_{\beta,R} e^{i(\phi_{\alpha,L}-\phi_{\beta,R})} \tag{2.97}$$

we can extract how much the phases of two distinct levels differ at a given time. In particular, we can follow two different procedures for the choice of the levels; we define:

$$z_\alpha(t) = \langle\Phi(t)|\, b^\dagger_{\alpha,L} b_{N/2,R}\,|\Phi(t)\rangle\ , \qquad w_\alpha(t) = \langle\Phi(t)|\, b^\dagger_{\alpha,L} b_{\alpha,R}\,|\Phi(t)\rangle \tag{2.98}$$

In the first case, the subscript refers to the level on the left grain and a reference state is taken on the right grain; conversely, with the other choice of correlation function, the level is chosen to be the same on both grains. The two are expected to produce the same phase only when the grains show coherent behaviour, which means that the phase difference between them is, within a good approximation, given by the phase difference between *any* two levels chosen.

Note that, on general grounds, the functions (2.98) are functions of *both* time and of the level index. It is then necessary to verify whether, *for most of the time*, levels have small phase difference. We start by the analysis of the mean $m_{w,z}(t)$ and the standard deviation $\sigma_{w,z}(t)$ of the level phases, at each given time, as computed from both w and z functions. The corresponding figures, for a model initial state, are shown for different pairing strengths in Figures 2.8 and 2.9.

The information at given time is useful, but not complete, as we need information about the time *evolution* of the system. In particular, to assign a unique phase difference to the grains, one needs assess whether a given value of g shows a satisfactory degree of phase variance when taking into account different couples of levels *for most of the time*. In order to give a

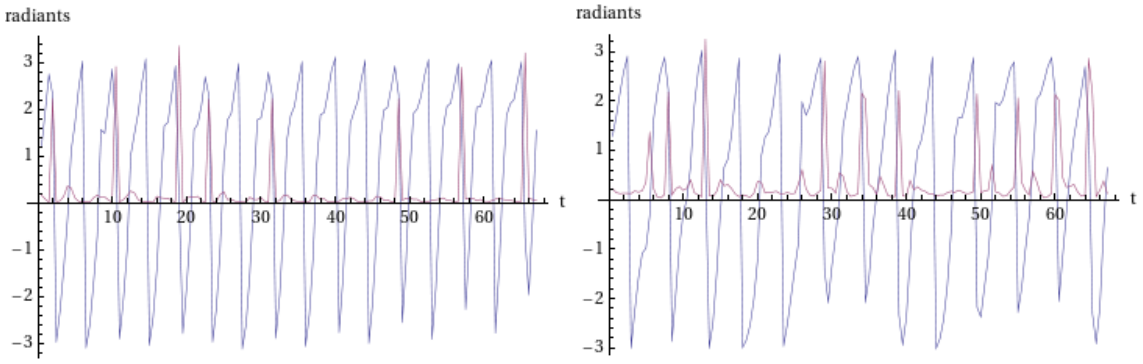


Figure 2.8: Phase difference mean (blue) and standard deviation (purple) with respect to level index, as determined from correlation functions z (right) and w (left), for two coupled grains with $N = 8$ levels each, total number of pairs $M_T = 8$, pairing strength $g = 0.6$, tunnelling parameter $\lambda = 0.1$ and initial unbalance $z_0 = 0.25$

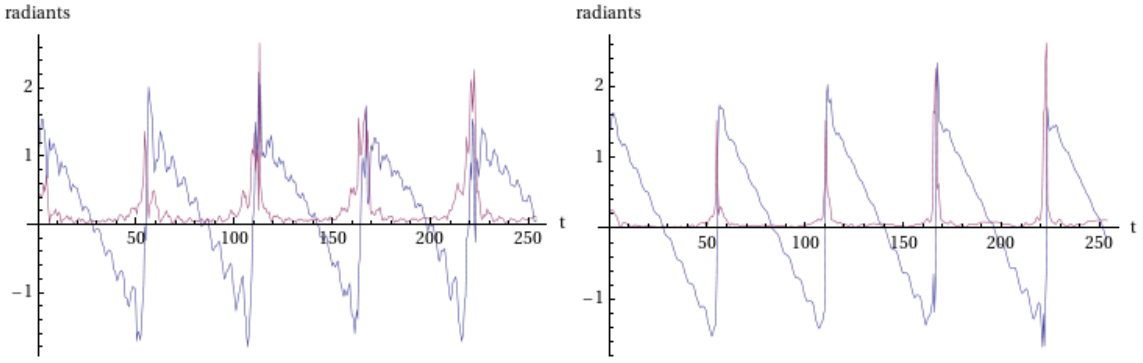


Figure 2.9: Phase difference mean (blue) and standard deviation (purple) with respect to level index, as determined from correlation functions z (right) and w (left), for two coupled grains with $N = 8$ levels each, total number of pairs $M_T = 8$, pairing strength $g = 0.2$, tunnelling parameter $\lambda = 0.1$ and initial unbalance $z_0 = 0.25$

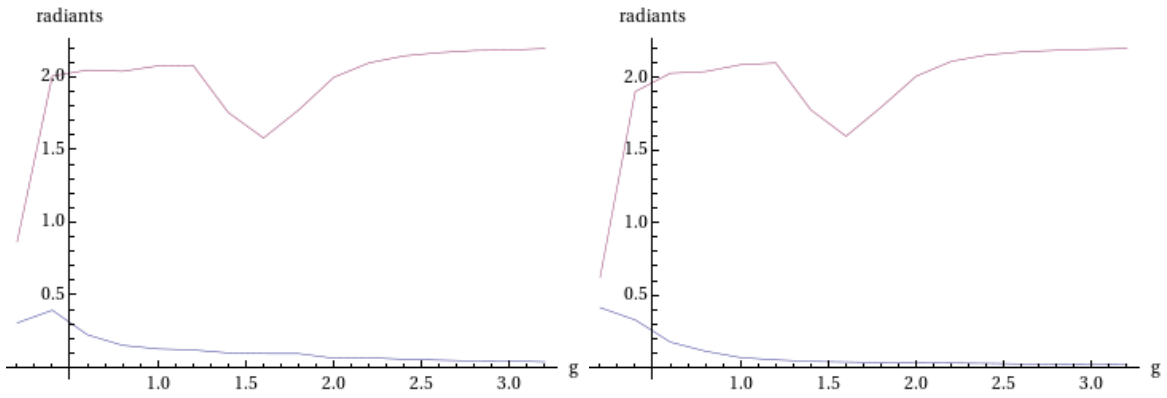


Figure 2.10: Left: parameters C_z^ϕ (blue) and S_z^ϕ (purple). Right: parameters C_w^ϕ (blue) and S_w^ϕ (purple), as defined in the main text. Grains have $N = 8$ levels each, total number of pairs $M_T = 8$, tunnelling parameter $\lambda = 0.1$ and initial unbalance $z_0 = 0.25$

more quantitative estimate of this property, we consider the mean of the standard deviation presented above over sufficiently long times (several periods) $C_{z,w}^\phi = \frac{1}{t}\int_0^t \sigma_{z,w}(t')dt'$. To establish a comparison, we need to evaluate also the magnitude of the mean phase difference among the condensates. This is a rapidly varying quantity, whose characteristic frequency is proportional the absolute value of the ground state energy. Since it has zero mean, we are interested in computing the characteristic range over which it varies, i.e., its standard deviation computed during the time evolution $S_{z,w}^\phi = \frac{1}{t}\int_0^t m_{z,w}^2(t')dt'$.

In Figure 2.10 we plot C^ϕ and S^ϕ as computed from corresponding levels on the two sides and by taking a reference level on one grain. As a function of the pairing parameter, the two procedures convey the same information: the highest the pairing, the more grains show coherent behaviour.

By applying this line of reasoning, we could verify some features about the phase relationship between grains for growing g. First, that the smaller the tunnelling parameter, the sooner (in g) a well-defined phase is established. Equally, that a small initial unbalance allows a definite phase to be built for relatively small values of g, while it is necessary to enforce stronger pairing if states with large initial unbalances are selected. This is due to the fact that the initial states are projected on few states in the lowest part of the spectrum when the initial population difference is small. Conversely, large population differences at $t = 0$ are projected to many states in the middle of the spectrum, each having its own energy.

An important aspect to consider is the presence of fermion tunnelling. As a matter of facts, the original Hamiltonian (2.51) is written in terms of fermionic operators, while the results obtained – that the phase coherent behaviour is established with relatively small pairing – refer to the bosonic approximation, acting on the restricted subspace. If, on the one hand, it is plausible that at vanishing g pair-breaking excitations may play an important role, a natural question to ask is whether the presence of fermionic degrees of freedom, aside of bosonic pairs, may spoil the phase-coherent behaviour of the grains for sufficiently large pairing. The issue can be rephrased into the question of whether the initial state, during the evolution generated by the coupled Hamiltonian, containing a fermionic tunnelling term, may give rise to a huge number of states in which two or more electrons are *not* paired, which evolve incoherently with respect to the states in which only pairs appear.

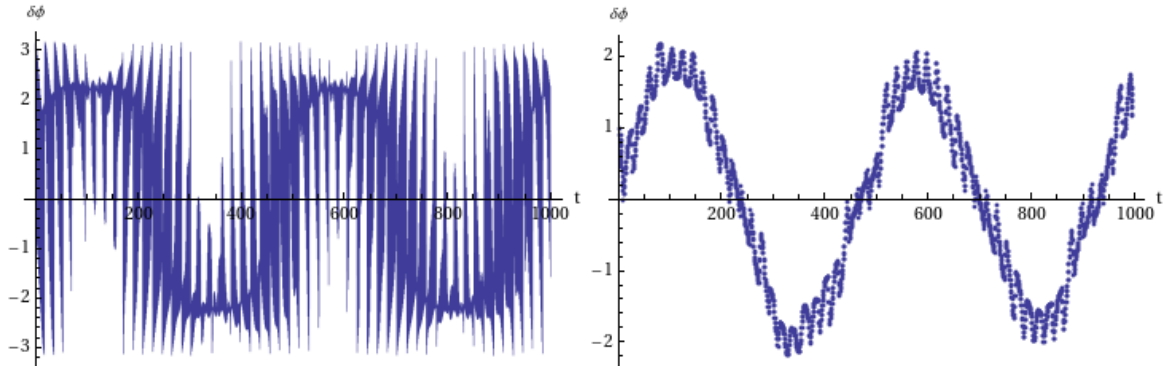


Figure 2.11: Left: phase difference, as a function of time. Right: averaging over short times removes fluctuations.

These states must be written as linear combinations of the factorized states of the two uncoupled Hamiltonians. On each grain, the energy of such states can be computed exactly for any value of g. In order to have an estimation of a lowest bound for the energy, we can consider a state in which the most energetic pair is broken and one electron is promoted into the next level, which reduces the number of pairs by 1 and the number of unblocked levels by 2, as seen in section 1.4. The energy of the lowest pair-breaking excitation has been considered in [33] and reads:

$$E_{pair} \simeq \frac{\epsilon_M + \epsilon_{M+1}}{2} - g(M-1)((N-2)-(M-1)+1) \tag{2.99}$$

The bare energies in the first term of the above do not depend on g, unlike the ground state energy, all the pair-conserving excitations and the second term in the previous equation. It follows that, by taking the pairing strength sufficiently high, all pair-breaking excitations can be made to lay at arbitrary energy above the ground state and are therefore suppressed with respect to pair-conserving excitations.

Checking explicitly that the insertion of states with unpaired electrons does not spoil the phase relation requires much larger computational effort, in that the Hilbert space should be enlarged to the $\binom{N}{m}\binom{N-m}{M}$ configurations in which the m electrons can "block" part of the N levels, with fixed number M of pairs. We therefore rely on the standard argument based on the presence of a gap. Note that this should already hold for values of $g \geq 0.25$, as discussed earlier.

We also mention that, even if the phase is quite well-defined, residual fluctuations can still be observed, in such a way that the widest, slowest oscillations are superimposed with faster and narrower ones. What is relevant are only the former ones, so that we find convenient to isolate them by computing time averages on intervals much smaller than the period of the largest oscillations: this allows to better understand the structure of the diagrams. An example of the procedure is provided in Figure 2.11.

2.4.5 Occupation–phase diagram and current–phase characteristic

In order to achieve the largest possible freedom in selecting the initial conditions, we imagine to prepare the two uncoupled leads at $t = 0$ in a linear superposition of the states in which the total number of pairs is fixed to be $M_T = N - 1$. Calling $\left|\Phi_{g,M}^{(L,R)}\right\rangle$ the lowest-energy state

ξ	δM
0.1	0.98
0.2	0.92
0.3	0.83
0.4	0.72
0.5	0.60
0.6	0.47
0.7	0.34
0.8	0.22
0.9	0.10

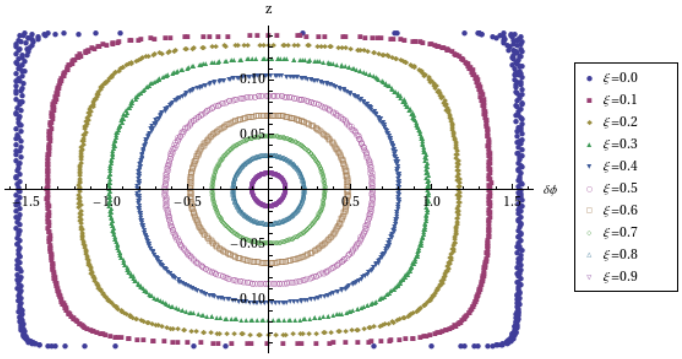


Figure 2.12: Phase diagram for different values of the parameter ξ in the BCS regime, with $N = 8$, $M_{TOT} = 7$, $g = 0.571$.

with M pairs of either the left or the right grain, we prepare the system in the state

$$|\Psi_0\rangle = \frac{1}{\sqrt{1+\xi^2}} \left(\left|\Phi^{(L)}_{g,M}\right\rangle \otimes \left|\Phi^{(R)}_{g,M-1}\right\rangle + e^{i\phi_0}\xi \left|\Phi^{(L)}_{g,M-1}\right\rangle \otimes \left|\Phi^{(R)}_{g,M}\right\rangle \right) \tag{2.100}$$

where the initial phase difference ϕ_0 and the initial population unbalance $\delta M_0 = \frac{1-\xi^2}{1+\xi^2}$ can be selected by choosing the corresponding parameters appropriately. We then turn on a small perturbation ($\frac{\lambda}{d} = 0.01 - 0.1$ in our runs) and compute the time evolution of the state after exact diagonalization the Hamiltonian. The main limitation of this protocol arises from the consume of RAM by diagonalization subroutines; by limiting subspaces appropriately, one can study systems of up to $N = 10$ levels, below half filling and for a small initial population unbalance.

The first, already nontrivial, issue we would like to check is whether it is possible to draw a population–phase diagram in the spirit of [57, 56] and if it can be seen to fit a two–level model. The confined phase can be explored by tuning the initial state to different particle unbalances. We report an example of the results in figure 2.12

The phase diagram in the plane ϕ, z shows a remarkable agreement with the "pendulum" law of motion in the small oscillations regime when the initial unbalance is small (see figure above). In addition to that, as the oscillations become more pronounced, we see that the motion receives important corrections. It is a characteristic of the BEC regime the fact that the phase does not overcome the value $\phi = \pi/2$ when the initial unbalance is of one particle, which agrees with the predictions of (2.79). This fact has been checked also with exact diagonalization and is certainly not and artefact the Hilbert space truncation. Instead, it is connected with the fact that the main contributions to the wavefunction arise from the first two lowest-lying states of the interacting system with equal weights. The very high coherence of the grains confirms this aspect.

It is also possible to study the diagram while varying the initial phase in the initial state (2.100). Also in this case, as shown in figure 2.13, the "stretched pendulum" with $\Lambda \to 0$ seems to agree with the figures. But it has to be remarked that this picture seems to hold only for the unbalance $\delta M = 1$. For larger values, the phase is allowed to oscillate further, and more complicate picture emerges.

We now focus on the pair current between the models, defined as the time derivative of

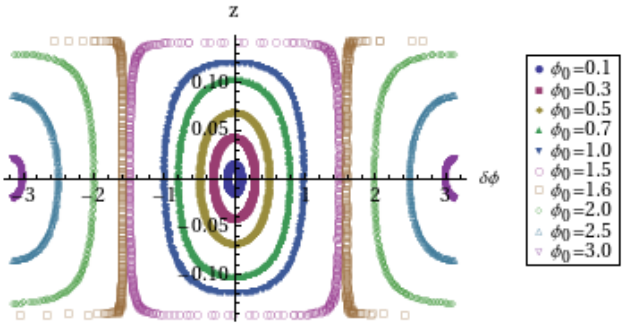


Figure 2.13: Phase diagram for different initial phases ($N = 8$, $M_{TOT} = 7$, $\delta M(0) = 1$, $g = 9.7$).

the occupation number of the left subsystem.

$$I(t) = i\,[H, N_L] = i\,[H_T, N_L] = i \sum_{\alpha,\beta} \frac{b_{\alpha,L} b^\dagger_{\beta,R} - b^\dagger_{\alpha,L} b_{\beta,R}}{\sqrt{\xi^2_{\alpha,L} + \Delta^2} + \sqrt{\xi^2_{\beta,R} + \Delta^2}} \tag{2.101}$$

From a computational perspective, we can easily evaluate the time evolution of the occupation number difference $\delta M = N_L - N_R$ and take the derivative numerically.

It is possible to verify, for $\delta M = 1$ that, in this configuration, the levels involved are mainly the first two, with minor contributions from the ones above. The frequency observed is then trivially the difference of the two and, as a function of g, goes to a constant value that characterizes the bosonic side.

One feature of the the evolution of the phase is that it never crosses the value $\delta\phi = \pi/2$. It follows that the current–phase characteristic can be fitted with a form

$$I(\phi) = I_c(g, \lambda) \tan \phi \tag{2.102}$$

The critical current is seen to have a maximum around $g \simeq 1$ and can be fitted in the form

$$I_c(g, \lambda) = I_0 \lambda \frac{e^{-c/g^2}}{g} \tag{2.103}$$

with $c \simeq 0.27$, nearly independent on λ. This relation has a maximum at $g^* = \sqrt{2c}$.

Please, note that it is possible to roughly fit these data in the two–mode model with $\Lambda = 0$. This must be true, as long as the two lowest levels are the ones mainly involved in the dynamics. The fact that the *noninteracting* model is reproduced arises from the linearity of the problem and from the fact that the two levels are well separated from the rest of the spectrum.

The dominant frequency ν, i.e., the Fourier component with the highest weight, can also be investigated by our methods. It turns out to be proportional to the critical current, therefore given by (2.103).

It is an interesting issue to explore what happens when more levels, inserted in a band structure as the one described above, participate to the dynamics: in this situation, the effect of the interaction should be evident. Additional care must be paid to check that the spreading

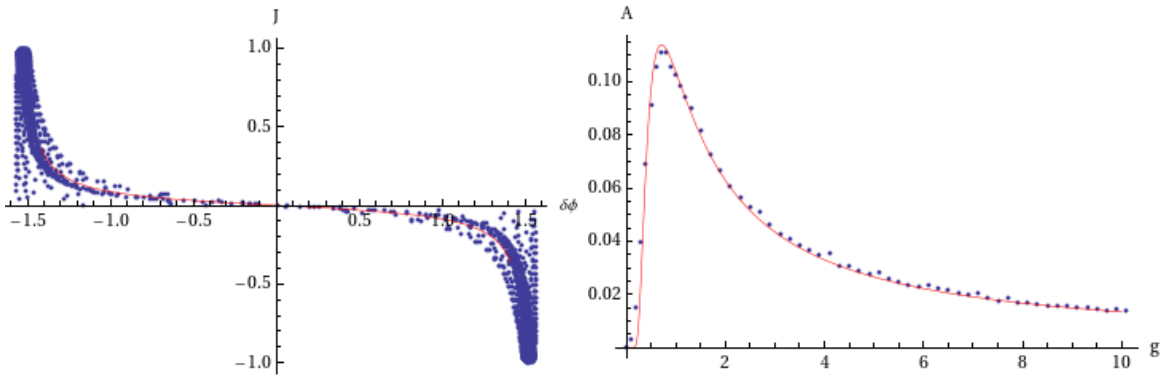


Figure 2.14: Left: tangent fit for $g = 2.3$, $\delta M(0) = 1$, $N = 8$, $M_T = 7$. Right: critical current fit with the law in the main text.

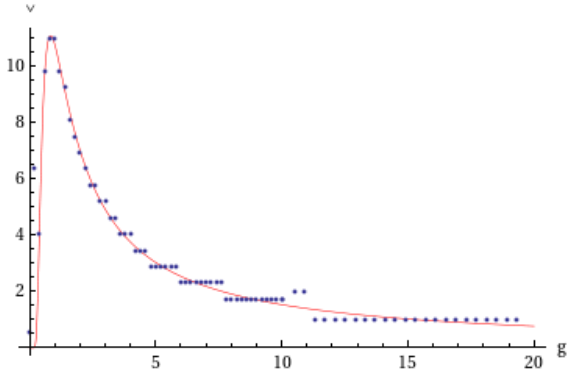


Figure 2.15: Dominant frequency $\delta M(0) = 3$, $N = 6$, $M_T = 4$.

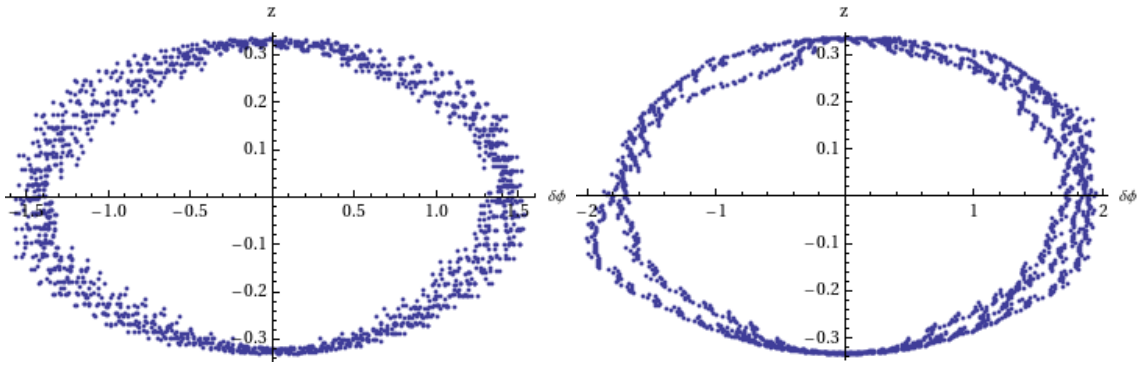


Figure 2.16: Occupation–phase diagram for $N = 8$, $\delta M(0) = 2$, $M_T = 7$ and $g = 0.4$ (left), $g = 6.4$ (right). The phase range increases with the pairing interaction.

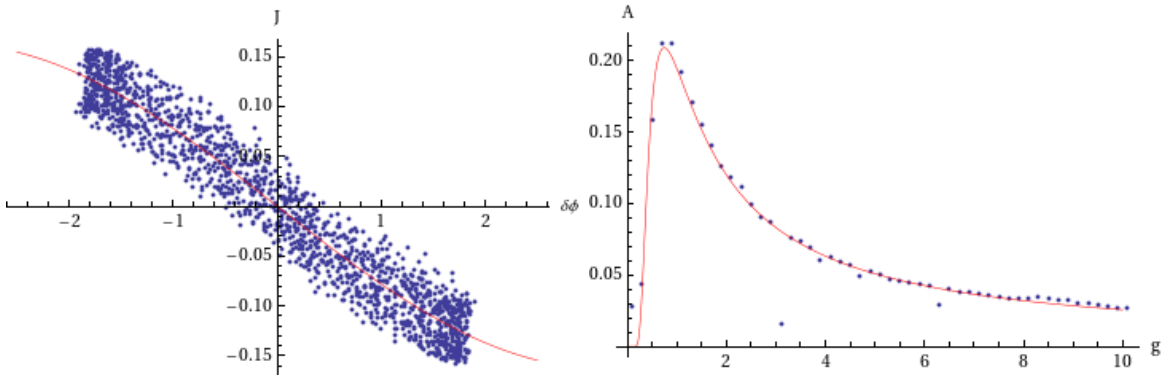


Figure 2.17: Left: sine fit for $g = 2.6$, $\delta M(0) = 1$, $N = 8$, $M_T = 7$. Right: amplitude fit with the law in the main text.

of the phases is constant: for this reason, we were not able to study particle unbalances overcoming the value $\delta M = 2$.

For what the phase diagram is concerned, we found that it fits the expectation from a two–mode model, showing the typical ellipsoid form. The phase range depends only on the interaction, while the amplitude of the population oscillations depends on the initial relative phase given to the system through (2.100).

However, no explicit trapping can be observed. What is observed, instead, is that the amplitude of the fastest oscillations is increased and that the period of the slowest decreased more and more, as $1/g$. The scenario is that of a large crossover to a confined regime, in which the occupation oscillations have infinite period at $g \to \infty$. This may be a finite–size effect, leading to a transition in the thermodynamic limit. The dependence of the frequency one the pairing strength is given by (2.103). Our finding is that the law is

$$I(\phi) = I_c(g, \lambda) \sin \frac{\phi}{2} \tag{2.104}$$

with the same dependence for the amplitude as in (2.103). The dependence on the phase was predicted for a weak, pointlike barrier in the WKB approximation in the Bogolubov-de Gennes equation [63]. Here we qualify the dependence on the pairing strength of the critical

current.

The initial phase can also be varied with initial unbalance $\delta M_0 > 1$. It is interesting to note that the for most of the values of g, the phase runs. Nevertheless, the time evolution of the mean phase locks it around some large–period oscillation. Quantitative evaluation of the phase diagrams remains a nontrivial task.

2.4.6 Conclusions

We have performed a numerical analysis of the exact dynamics of two coupled Richardson Hamiltonians, with attractive interaction and fixed level spacing, initially prepared in the ground state of the unperturbed system. From the behaviour of the chemical potential, we suggested that the model can describe the crossover the BCS and BEC regimes and shown that these regimes are clearly distinguishable by the spectrum of the coupled models. We have explained a criterion for assessing the formation of a definite phase relation and found, for the cases in which the coupled models showed coherent behaviour, that a unique relation connects the occupation number and the current with the phase difference, throughout the crossover.

In the strongly coupled regime, one expects that an effective nonlinear dynamics for the condensate density would emerge. Nevertheless, we are dealing with a linear Hamiltonian, which couples a set of discretely–spaced levels. One way of seeing the effective nonlinearity is in the highly–degenerate banded structure that is created when $g \to \infty$: a given initial state, eigenstate of the unperturbed Hamiltonian, is written as a linear combination of states in different bands. When the occupation unbalance at $t = 0$ is equal to one, the behaviour is well described by a noninteracting two–mode approximation, since the involved levels are mainly two and are well-separated by all the others. Whenever the levels involved are more, one has to consider the effect of the band structure. With more than two bands, a multi–mode level may be needed to describe the effective many–body dynamics.

It stands as an open question how our findings scale with the size of the system. This was mainly due to the need of diagonalizing and multiplying large matrices, which is both time- and memory- consuming. This is an important issue, which we intend to investigate in the future.

Chapter 3

Many-body localization in the Richardson model

Recently [64, 65] it has been pointed out that the phenomenon of Anderson localization [66], usually associated with single-particle hopping in a random potential, can be present even in the many-body eigenstates of an interacting quantum system and manifest itself as a phase transition at finite or infinite temperature.

This phenomenon has been dubbed *many-body localization* (MBL) and can be conceived as an example of Anderson Localization (AL) on configuration space, rather than real space. As the geometry of configuration space for a many-body system is quite different from that of a regular lattice in few dimensions, many-body localization is thought to have properties distinct from those of the single-particle localization.

This phenomenon should be responsible, among other things, of the exact vanishing of the DC conductivity of metals below a critical temperature [64] and of the failure [67] of the simplest version (and possibly of all versions) of the quantum adiabatic algorithm [68] for the solutions of NP-complete problems; it has also been studied in disordered Heisenberg spin-chains [69, 70] where the phase transition has been linked to the infinite-randomness fixed point. The similarity of some features of many-body localization to the glass transition in spin and configurational glasses makes it the closest to a *quantum* analog of a glass transition, where the assumptions of equilibrium statistical mechanics fail.

As we said, in some problems, many-body localization is found in typical many-body states[71], namely states sampled with uniform distribution from the spectrum (therefore corresponding to infinite temperature). These states are difficult to study directly, much more than the ground states for which many approximations (DMRG, MPS etc.) can be devised: indeed the only strategy here seems to be exact diagonalization (as used in [70] for example), the exponential complexity of which limits the size of the systems to less than 20 spins; alternatively the study of correlation functions with time-dependent DMRG was used, whose failure to converge due to growing entanglement can signal the onset of delocalization[72].

In this chapter, an introduction to many-body localization is provided in section 3.1. An original contribution is illustrated in section 3.2, where the disordered version of the hamiltonian (1.86) is studied, focusing on the localization properties of the exact many-body eigenstates. As it will be made clear below, the possibility of computing exact matrix elements between eigenstates of hamiltonians, as well as exact overlaps, even for different values of the interaction, turns out to be essential in our approach.

3.1 Localization in the Fock space

3.1.1 Hopping conduction

A single-particle wavefunction $\phi(\vec{r})$ in a d-dimensional disordered potential may have different properties, with respect to the spatial extension of its support. In particular, for an eigenstate, a major role is played by the strength of the disorder. To be specific, the eigenstate can be either *localized*, when peaked around some point in space and exponentially decaying away from that point:

$$|\phi|^2 \propto \frac{1}{\xi^d} e^{-\frac{\vec{r}-\vec{r}_0}{\xi}} \tag{3.1}$$

with ξ the localization length, or *extended*, when

$$|\phi|^2 \sim \frac{1}{size} \tag{3.2}$$

in words, when there is a nonvanishing probability of finding the particle approximately at every point in space.

It is currently accepted (see [73]) that localized and extended states cannot coexist at the same energy, so that the spectrum splits into bands of localized and extended states, separated, in $d \geq 3$, by a *mobility edge* at energy $\mathcal{E}$. The presence of such a feature in the spectrum implies that conduction is provided only by the states which have enough energy. This leads to a conductivity of the form

$$\sigma \propto e^{-\frac{\mathcal{E}-\epsilon_F}{k_B T}} \tag{3.3}$$

known as *activated* conductivity, in which only states above a certain energy, which are present in the thermal mixture with an exponentially suppressed amplitude, conduct.

In the limit of very strong disorder, when all single-particle eigenstates around the Fermi level are localized, it is possible to model our sample as a collection of sites, which are characterized by a set of randomly-distributed single-particle eigenstates. These sites are distributed in a regular lattice and some mechanism allowing hopping from one site to the other is needed for conduction.

One possible mechanism in that of thermal activation, if the density of states at the Fermi energy ν is finite. The energy which is necessary to hop from one localized state to the other can be provided by a bath, in equilibrium at some temperature T. The details of the bath are not essential and one just requires continuous spectrum of delocalized excitations down to zero energy. One example of such bath is a *phonon* bath [74]: due to the continuity of the spectrum, any energy mismatch can be absorbed by a phonon of appropriate energy. As a consequence, $\sigma(T)$ turns out to be finite, although very small, at any finite T even when all one-electron states are localized.

As described in [75], if the electron is supposed to move always to the nearest empty site, the temperature dependence of the conductivity would still be of the form (3.3), in which the energy appearing is that of the state. However, it was pointed out in [74] that at low temperature the dominant process would not be to a nearest neighbour. In facts, within a range L from a given site, the density of states per unit energy range near the Fermi energy is

$$N_L = \Omega_d L^D \nu \tag{3.4}$$

where $\Omega_D = 2\pi^{D/2}/\Gamma(D/2)$ is the angular integral in dimension D. It follows that for the hopping process through a distance L with the lowest activation energy, this energy mismatch

would be the reciprocal of this number $\delta E = 1/N_L$. Then, the further the electron hops, the smaller will be the energy required for the process.

However, hopping over a large distance involves a tunnelling whose probability is suppressed by the factor

$$e^{-2L/\xi} \tag{3.5}$$

where ξ is a localization length characterizing the the decay of the localized wavefunction. There will be an optimum hopping distance L, maximizing the probability

$$e^{-2L/\xi - \frac{\delta E}{k_B T}} \tag{3.6}$$

which happens when the exponent has its minimum value, i.e., when

$$L = \sqrt[D+1]{\frac{\xi D}{2\,\Omega_D\,\nu\,k_B T}} \tag{3.7}$$

This model gives the conductivity of the system to be

$$\sigma = \sigma_0 e^{-B_d T^{-1/(1+D)}} \tag{3.8}$$

in which $B_D \propto \left(\frac{2^D}{xi^D \Omega_D \nu}\right)^{\frac{1}{D+1}}$ and σ_0 has some weak (power law) dependence on temperature. Together with T_0, it depends on the details of the model. This form takes the name of Mott's *variable range* conductivity.

Even when the dominant mechanism of conduction is provided by the phonon-mediated hopping, but one has many particles on many accessible levels within a small region, the degree of localization of an electron on the levels at a specific site modifies the energy mismatch which has to be overcome in the transition. Somewhat oversimplifying, if one assumes a uniform delocalization over the whole local configuration space, it is reasonable to expect that the highest tunnelling rate will occur for the couple of levels which exhibit the smallest energy difference. This process accesses an energy mismatch which is given by

$$\delta E = \frac{1}{\left(\Omega_D L^D \nu\right)^2} \tag{3.9}$$

Then, by repeating the above arguments, one finds that the conductivity (3.8) is modified to:

$$\sigma = \sigma_0 e^{-B_D T^{-\frac{1}{2D+1}}} \tag{3.10}$$

with $B_D = \left(\frac{2^D}{\xi^D \Omega_D \nu}\right)^{2/(2D+1)} \left((2D)^{1/(2D+1)} + (2D)^{2D/(2D+1)}\right)$. One sees that in the two extrema – on each site either only one level or the whole set of accessible levels are occupied – correspond to two different temperature dependence of the conductivity. This drives the interest toward the presence of a many-body localization *transition* on the local space and in the possibility of intermediate situations, in which the wavefunction does not cover the whole local space.

In this case, the "local" space is constituted by the set of levels within the *single-particle* localization length. On the other hand, whenever within the same range many electrons are present, the *interactions* among them imply that the many-body wavefunction may be expressed as a linear combination of several many-body eigenstates of the noninteracting problem, the latter being labelled by the single-particle level occupation numbers. As a result, the interaction strength drives the degree of *many-body* delocalization, playing a role in the hopping conduction process. The simplest approximation for the local hamiltonian is provided by (1.86), where a uniform interaction couples a set of otherwise independent, randomly distributed levels.

3.1.2 The role of interactions in conduction

A question which is legitimate to ask is whether an interaction between electrons on the same site may be an efficient mechanism for hopping conduction, even in the absence of a phonon-mediated hopping, at low temperatures. The issue has been explored in [64], where a non-trivial answer was found. Below a critical temperature, conductivity is exactly vanishing, while above it, interactions can actually determine hopping efficiently and the behaviour is that of a metal. The mechanism at the roots of this peculiar behaviour is many-body localization.

To be more precise, an hamiltonian describing a simple system of spinless electrons with two-body interactions is given by

$$H = \sum_{\alpha} h_{\alpha} c_{\alpha}^{\dagger} c_{\alpha} + \frac{1}{2} \sum_{\alpha,\beta,\gamma,\delta} V_{\alpha\beta,\gamma\delta} c_{\alpha}^{\dagger} c_{\beta}^{\dagger} c_{\gamma} c_{\delta} \tag{3.11}$$

with matrix elements:

$$V_{\alpha\beta,\gamma\delta} = \int d\vec{r} \int d\vec{r}' \phi_{\alpha}(\vec{r})^* \phi_{\beta}(\vec{r}')^* V(\vec{r} - \vec{r}') \phi_{\gamma}(\vec{r}') \phi_{\delta}(\vec{r}) \tag{3.12}$$

where ϕ_{α} stands for the single-particle wavefunction relative to level α. These levels are, by definition, eigenstates of the hamiltonian in the absence of the interaction term.

Consider a weak, short-range interaction of the form

$$V(r_1 - r_2) = \frac{\lambda}{\nu} \delta(r_1 - r_2) \tag{3.13}$$

where λ is some generic dimensionless parameter, controlling the interaction strength and the δ function means simply that the decay of the potential is exponential and the range is much smaller than the electron mean free path, with some characteristic scale given by the localization length ξ.

Following the exposition of [65], the relevant energy scale in a problem where a the wavefunction is exponentially decreasing away from a given site, with localization length ξ, is the energy spacing between states localized nearby, which can be written as

$$\delta_{\xi} = \frac{1}{\nu \xi^d} \tag{3.14}$$

where ν is the one-particle density of states. We can moreover write the typical value of the interaction matrix element as $V \simeq g\delta_{\xi}$. In the many-body problem, one has that the energy mismatch of a virtual two-body transition is given by $|h_{\alpha} + h_{\beta} - h_{\gamma} - h_{\delta}|$, which has typical value δ_{ξ}.

The matrix element (3.12) embodies the process of the decay of a fermion in the state α, which is one single-particle excitation, into two fermions in the states γ and δ, plus an hole in the state $\bar{\beta}$. During time evolution, under the action of (3.13), newly produced single-particle excitations (electrons injected in the sample) will decay on their turn into more and more single-particle excitation.

$$|\alpha\rangle \to \left|\bar{\beta}, \gamma, \delta\right\rangle \to |\bar{\varepsilon}, \bar{\sigma}, \omega; \rho, \ldots\rangle \to \ldots \tag{3.15}$$

The actual weight of the newly-created excitations into the resulting many-body wavefunction depends on the matrix elements (3.12). In particular, a *localized* state is the superposition

of very few quasiparticle states, which possess a sufficiently well-defined energy, so that the initial electron is never completely decayed by adding orders in perturbation theory. On the other hand, *delocalized* states possess components with higher and higher number of single-particle excitations, and the original single-particle excitation spreads irreversibly (in an infinite system) onto all the accessible many-body states.

The issue can be looked also from the perspective of the quasiparticle spectral function

$$A(\epsilon)_\alpha = \sum_k \left| \langle \Psi_k | \, c_\alpha^\dagger \, | \Psi_0 \rangle \right|^2 \delta(\epsilon + E_0 - E_k) \tag{3.16}$$

seen as a function of the interaction strength. In facts, the eigenstates of the interacting system can be obtained in perturbation theory, which in turn defines a perturbative expansion for the spectral function

$$A(\epsilon)_\alpha = \sum_n A_\alpha^{(n)} \lambda^n \tag{3.17}$$

The initial bare particle peak corresponds to $A^{(0))}$. Each order in perturbation theory will spread this peak either further and further, or conversely on nearby energies, retaining the shape of a well-defined resonance.

A result corresponding to a *localization in the many-body Fock space* was found in [64] for the hamiltonian (3.11), by using many-body perturbation theory. Two phases where found: a *metallic* and an *insulating* phase. A metal-insulator transition was shown to take place at

$$T_c = \frac{\delta_\xi}{C_2 g \log(1/g)} \tag{3.18}$$

where C_2 is a model-dependent constant.

Consider a many-body eigenstate $|\Psi_j\rangle$ of the full hamiltonian (3.11). If one creates an electron-hole pair on top of this state, the outcome will not be an eigenstate any more, but may still be expanded in terms of all the other eigenstates as

$$|\Psi_{j;\alpha\beta}\rangle = c_\alpha^\dagger c_\beta \, |\Psi_j\rangle = \sum_k C_{j,k}^{\alpha,\beta} \, |\Psi_k\rangle \; ; \quad \sum_k |C_{j,k}^{\alpha,\beta}|^2 = 1 \tag{3.19}$$

The quantity characterizing the localization of a state $|\Psi_{j;\alpha\beta}\rangle$ (on the "computational" basis or configuration space $\mathcal{C}$) is the *inverse participation ratio*:

$$\mathcal{I} = \left(\sum_{s_1,...,s_N} | \, \langle s_1, ..., s_M | \Psi_{j;\alpha\beta} \rangle \, |^4 \right)^{-1} \tag{3.20}$$

where $s_1, ..., s_M$ is the configuration of the noninteracting system, either given in terms of spins or of level occupation number. An *insulating* phase corresponds to *localized* many-body states, such that the inverse participation ratio does not diverge in the thermodynamic limit

$$\lim_{size \to \infty} \frac{1}{\sum_k |C_{j,k}^{\alpha,\beta}|^4} < \infty \tag{3.21}$$

which signals that the excitation cannot propagate over all states. Conversely, when an infinite number of eigenstates enters the expansion (3.19)

$$\lim_{size \to \infty} \frac{1}{\sum_k |C_{j,k}^{\alpha,\beta}|^4} = \infty \tag{3.22}$$

we call the many-body state *extended* and the corresponding phase *metallic*. This is not equivalent to *ergodicity* within a given energy shell, since for the latter property to occur, one needs the many-body wavefunction to be almost uniformly delocalized over the *whole* energy shell and not simply over an extensive subspace. Then the expectation value over the exact many-body state is equivalent to averaging over the microcanonical distribution around the state energy E.

Assuming thermal equilibrium can be done only in the latter scenario, in the thermodynamic limit. In this situation, the definition

$$C_V = \left(\frac{\partial E}{\partial T}\right)_V \tag{3.23}$$

holds and can be used to connect the energy of a many-body state to a temperature. In particular, the critical temperature T_c in (3.18) corresponds to a *mobility edge*, above which states are conductive and below which are insulating. An important difference with respect to the single-particle-localized scenario discussed above is that the dependence of conductivity from the temperature is different, when taking the infinite volume limit. In facts, assuming a Gibbs distribution at temperature T for the eigenstates, we have:

$$\sigma(T) = \sum_k P_k \sigma(E_k) = \frac{\int dE \sigma(E) N(E) e^{-E/T}}{\int dE N(E) e^{-E/T}} \tag{3.24}$$

where $N(E)$ is the number of states at a given energy E, i.e., by definition $N(E) = e^{size \times s(E)}$ where $s(E)$ is the intensive Boltzmann entropy. The temperature defines a dominant energy $E(T)$ via the saddle-point approximation of the integral, when the size of the system tends to infinity. It follows from the vanishing of conductivity of localized states that in an infinite system

$$\sigma(T) = \begin{cases} \sigma(T) = 0 & E(T) < \mathcal{E} \\ \sigma(T) = \sigma(E(T)) & E(T) < \mathcal{E} \end{cases} \tag{3.25}$$

which represents a striking feature of the conduction mechanism.

To conclude, the possibility of studying some version of (3.11) exactly is of extreme interest. In this perspective, the Richardson hamiltonian is a simple version of the one studied in [64], where all the matrix element between levels are equal to $g\delta_\xi \sim g/N$. This is the scaling which will be adopted in 3.2.

Thermalization

Note that the above considerations can be rephrased into an analogue analysis about the *time evolution* of a weakly interacting system after the interaction g has been suddenly turned on at, say, time $t = 0$. The time evolution of the expectation value of some observable $\mathcal{O}$ soon after the switching on, by the aid of the Zassenhaus formula, can be written as:

$$\begin{aligned} &\langle \Psi_0 | \, e^{i\,(H_0 + g\,H_I)\,t} \, \mathcal{O} \, e^{-i\,(H_0 + g\,H_I)\,t} \, | \Psi_0 \rangle = \mathcal{O} + i \, \langle \Psi_0 | \, [H_0 + g H_I, \mathcal{O}] \, | \Psi_0 \rangle \, t \\ &- \frac{1}{2} \, \langle \Psi_0 | \, [H_0 + g H_I, [H_0 + g H_I, \mathcal{O}]] \, | \Psi_0 \rangle \, t^2 \\ &- \frac{i}{3!} \, \langle \Psi_0 | \, [H_0 + g H_I, [H_0 + g H_I, [H_0 + g H_I, \mathcal{O}]]] \, | \Psi_0 \rangle \, t^3 + O(t^4) \end{aligned} \tag{3.26}$$

As time goes by, higher and higher orders in the interaction become relevant, and the resulting state will be a more and more complicate superposition of states which are further and

further away (in the Hilbert space) from the original eigenstate. After long enough time, a sort of "ergodic" behaviour may effectively emerge and one can therefore expect that the delocalization/localization properties may be connected with the appearance of thermalization after a quantum quench of its absence [76], when the system size goes to infinity.

3.1.3 A large critical region

The problem of localization on the Fock space is also intimately connected to the lifetime of single-particle excitations over the many-body ground state, as discussed in [77]. This paper pictures the many-body delocalization process as a diffusion on a simplified Hilbert space and provides a convenient framework for the analysis of the spreading in the Fock space of the exact many-body eigenstates, as resulting from the particles interaction.

Given the M-particle vacuum $|M\rangle$ of the free hamiltonian, obtained by filling the first M levels, one can consider all the Slater determinants constructed by creating m holes and m particles in the state

$$\Psi_M = c^\dagger_{\alpha_m} \dots c^\dagger_{\alpha_1} c_{\beta_m} \dots c_{\beta_1} |M\rangle \tag{3.27}$$

where the βs are below and the αs above the Fermi level. This state has an energy which is given by the sum:

$$E_{\Psi_M} = \sum_{j=1}^{m} \left(\epsilon_{\alpha_j} - \epsilon_{\beta_j}\right) \tag{3.28}$$

and can be represented by a string of zeros and ones to label the occupation number n_α of the states.

$$\Psi_M = |0, \dots, 0, 1_{\alpha_m}, 0, \dots, 0, 1_{\alpha_1}, 0, \dots, 1_F, 1, \dots, 0_{\beta_m}, 1, \dots, 1, 0_{\beta_1}, 1 \dots\rangle \tag{3.29}$$

The *Hamming distance* is defined on the many-body Hilbert space as the number of labels which differ in the two configurations:

$$d\left(\Psi, \Psi'\right) = \sum_\alpha \left(n_\alpha(\Psi) - n_\alpha(\Psi')\right)^2 \tag{3.30}$$

It follows that the interaction term in (3.11) connects only states at distance $d = 0,\, 2,\, 4$.

It is useful to think to the Hilbert space as divided in *generations*, arising from the hierarchical processes described in (3.15). Given the ground state with $M-1$ particles $|M-1\rangle$, all states of the form $b^\dagger_\alpha |M-1\rangle$ with one particle in the state α added to the ground state, are in the first generation, while those of the form $b^\dagger_\alpha b^\dagger_\beta b_\gamma |M-1\rangle$ are in the third generation and generation 5 is composed by all states like $b^\dagger_\alpha b^\dagger_\beta b^\dagger_\gamma b_\delta b_\epsilon |M-1\rangle$. It is essential that any state from generation $n+1$ is connected by (3.11) only to those states within the same generation or in generations $n+3$ or $n-1$.

We now add a particle over the ground state with energy $\varepsilon > \epsilon_F$, therefore obtaining a state of generation 1. A quick computation shows that the density of states coupled to the starting one in generation 3 with the same energy is

$$\nu_3(\epsilon) = \int_0^\infty \frac{d\,\epsilon_1}{\mathrm{d}} \int_0^\infty \frac{d\,\epsilon_2}{\mathrm{d}} \int_{-\epsilon_F}^0 \frac{d\,\epsilon_3}{\mathrm{d}} \delta(\epsilon - \epsilon_1 - \epsilon_2 + \epsilon_3) = \frac{\epsilon^2}{2\,\mathrm{d}} \tag{3.31}$$

with d being the quasiparticle level spacing. Iterating the process (taking care that states of further generations are unconstrained provided $n < \sqrt{\epsilon\,/\,\mathrm{d}}$) analogous considerations show

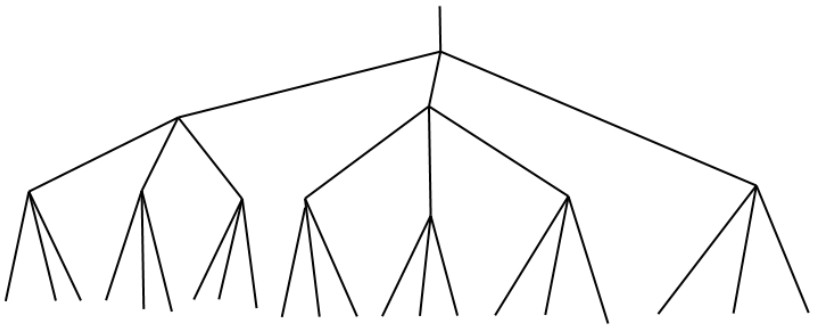


Figure 3.1: Bethe lattice with connectivity $K = 3$.

that for the $(n+1) - st$ generation the density of states grows like $\frac{1}{(2n)!}\frac{\epsilon}{2\,\mathrm{d}}$. Note also that the number of particle is conserved throughout all the hierarchy, which represents a major simplification in computational approach.

However, in order to split this contribution in elementary steps, we focus on the contribution to the amplitude of a state in generation $(2n+1)$ stemming from the *direct* access from a *given* state of generation $(2n-1)$, which is given by $\nu_{2n+1} = \nu_3/n$, as only one particle can decay out of n. The starting state under the action of H_I may also annihilate one of the n particles with one of the $n-1$ holes and jump back to the previous generation, with associated density of states $n(n-1)(2n-3)/\Delta$, or stay within the same generation with an associated density of states ϵ/Δ^2. For sufficiently small n, the latter are considerably smaller than the "forward" contribution.

To sum up, looking to the effect of interactions as a succession of elementary contributions on a given starting state and selecting only the most relevant ones yields a picture of the process as taking place on a Bethe lattice, in which each site is a many-body eigenstate, identified by an occupation configuration. From each site of generation $(2n-1)$, the system can jump to the $(2n+1)$-th generation.

Please note that this analysis, up to now, is valid for the fairly general hamiltonian of spinless electrons of the previous section. The somewhat drastic, yet useful, approximation that allows the problem to be treated is that of assuming all the matrix element in (3.11) to be equal $V_{\alpha\beta,\gamma\delta} \sim g$, which brings the hamiltonian (1.86) in the game.

Let us refine the predictions of perturbation theory for the Richardson model with coupling g/N. Let us start at $g = 0$ from state a, with energy E_a. The states at distance 2 from a have energies

$$\Delta^{(2)} = h_\alpha - h_\beta \tag{3.32}$$

where the couple $(\alpha, \beta) \in \mathcal{S}_\uparrow \times \mathcal{S}_\downarrow$ defines the spins which have been flipped up and down in going from a to b. The typical value of $\Delta^{(2)}$ is $\sqrt{\left\langle (\Delta^{(2)})^2 \right\rangle} = \sqrt{2} = O(1)$ however the

minimum value is $O\left(N^{-2}\right)$ which we write $x^{(2)}/N^2$ where $x^{(2)} = O\left(1\right)$. So the corresponding term in perturbation theory for the wave function is

$$A_b = \frac{(g/N)}{x^{(2)}/N^2} = \frac{gN}{x^{(2)}}. \tag{3.33}$$

In this way we can go on at arbitrary distance $2n$, to the state b^n, the amplitude thus having n denominators of $O\left(1/N^2\right)$

$$A_{b^n} = \frac{(gN)^n}{x^{(2)}x^{(4)}...\ x^{(2n)}}, \tag{3.34}$$

where x are random variables of $O\left(1\right)$. For any given a there are only $O\left(1\right)$ neighboring states with $\Delta \sim 1/N^2$, so the number of such b^n states at distance $2n$ from a is $O\left(1\right)$ out of N^{2n} (also the number of relevant paths does not grow as $n!$). These can be called a *direct* or *percolating* contribution. However already at distance 4 we observe another type of contribution, which one is tempted to dub a *tunnelling* contribution, in which although the final denominator $\Delta^{(4)} = h_\alpha - h_\beta + h_\gamma - h_\delta = z^{(4)}/N^4$ each of the two paths leading to the minimum $h_\alpha - h_\beta \simeq -(h_\gamma - h_\delta) = y^{(2)} = O\left(1\right)$, where $\alpha, \gamma \in \mathcal{S}_\uparrow$ and $\beta, \delta \in \mathcal{S}_\downarrow$. Again this contribution is of order:

$$A_b = \frac{(g/N)}{y^{(2)}} \frac{(g/N)}{z^{(4)}/N^4} = \frac{(gN)^2}{y^{(2)}z^{(4)}}, \tag{3.35}$$

while the amplitudes corresponding to the distance-2 intermediate steps are $O\left(1/N\right)$. The distribution of x, z can be found by using the theory of extreme value statistics [78], while y's are typical values of field differences and none of these distribution depends on N. We will stop here our analysis of perturbation theory as this would require a separate work by itself. It is sufficient for us to notice that only the combination gN appears in all terms of the series.

In our fully connected model, we have a well-defined connectivity, which is

$$K = \left(\frac{\epsilon_F}{2\,\mathrm{d}}\right)^2 \tag{3.36}$$

at half filling.

The problem of estimating the degree of localization of quasiparticle on the Fock space is then reduced to the localization on a Bethe lattice of a many-body state. This has been studied in [79] through the computation of the behaviour of the particle self-energy in the thermodynamic limit. The answer provided in the case of uniformly distributed on-site energies in the interval $[-W, W]$[1] was that there exist a localization-delocalization transition determined by the condition

$$\frac{K\,g}{W} \log K \sim 1 \tag{3.37}$$

Instead of considering the original derivation, one has a clear picture by focusing on the amplitude, at given energy ϵ, connecting a site of the first generation to site in the $(2n+1)$-th a matrix element g. The term (3.34) is the biggest amplitude that connects a given initial site to one site in generation $2k+1$. For a generic state in the same generation, the amplitude reads:

$$A_n = \prod_{j=1}^{n} \frac{g}{\epsilon - \epsilon_j} \tag{3.38}$$

[1] computations for any other distribution are not so simple. This provides anyway a reasonable estimate also in other cases.

In [77], the probability that this quantity is significant after n steps is $p(|A_n| > C)$, where C is some arbitrary $O(1)$ threshold value. One can define

$$\log |A_n| = n \log \frac{g}{W} + Y_n \tag{3.39}$$

with $Y_n = \sum_{j=1}^{n} y_j$ and $y_j = \log \left| \frac{W}{\epsilon - \epsilon_j} \right|$. Under the assumption that energies are independent and identically distributed random variables, the probability distribution for the y_js is

$$P(y_j) = e^{-y_j} \tag{3.40}$$

from which the probability for Y can be obtained and consequently for the amplitude

$$P(|A_n|) = \frac{g/W}{(n-1)!} \frac{1}{|A_n|^2} [\log (|A_n|(W/g)^n)]^{n-1} \tag{3.41}$$

Then the probability that the module of A_n exceed some value $C \sim 1$, in the case of weak interactions, is

$$p(|A_n| > C) \simeq \frac{1}{(n-1)!} \frac{1}{C \log \frac{CW^n}{g^n}} \left[\frac{g}{W} \log \frac{W^n C}{g^n} \right]^n \tag{3.42}$$

from which it follows that the probability that there are *no direct paths* between the original site to any of the states in the $(2n+1)$-th generation is

$$e^{-f_n} = (1 - p(|A_n| > C))^{K^n} \tag{3.43}$$

When the probability is much smaller than one and $n \gg 1$, (3.42) provides a criterion of localization by considering the behaviour of

$$f_n \simeq \frac{1}{\sqrt{2\pi n} C} \frac{1}{\log(W/g)} \left[\frac{gKe}{W} \log \frac{W}{g} \right]^n \tag{3.44}$$

If the expression above increases at large n, then for far enough generations one has $f_n \gg 1$, signalling a large "flux" of amplitude away from the first generation. Conversely, small values of (3.44) mean localization in the Hilbert space. A transition between the two scenarios takes place when the last factor in the previous equation takes unit value, which determines a critical g as a function of the effective connectivity (3.36) and of the noise width

$$\frac{g^{**} Ke}{W} \log \frac{W}{g^{**}} = 1 \tag{3.45}$$

In addition to this, we can focus only on a one-step contribution to the diffusion: the condition that *all* generations are well connected with the first is

$$f_1 = \frac{K\, g}{W\, C} > 1 \tag{3.46}$$

, which provides another critical value g^* for the coupling.

An articulated picture arises at this point. In the localized phase $g < g^*{*}$, the first generation is weakly connected with the rest of the network. Therefore, the exact eigenstates are written as a superposition of few occupation configurations; similarly any injection of a further electron (or flip of a spin) will overlap with few exact eigenstates. For $g > g^*$, there are significant trajectories that connect all the generations and we call the states *delocalized*. This

may still not be sufficient to give *ergodic* behaviour, since only a portion, although extensive, of the Hilbert space may be covered. For intermediate regions $g^{**} < g < g^*$, there are paths that connect generation 1 to any generation, yet eigenstates of the noninteracting problem are connected to few states in the various generations. For a given realization of the energy levels, a finite number of generations is connected to the initial state. Moreover, different initial states will be arguably spread onto different states of any given generation.

Our work, in the following section, will show that the the the second scenario is the one met for the physical scaling of the interaction as $1/N$, where the condition $g > g^*$ is satisfied. In particular, despite being the spreading on the Hilbert space extensive in the size of the system, only a small portion of it is occupied by typical states.

3.2 Numerical study

The Richardson model (1.86) can accommodate quenched randomness in the arbitrary choice of the fields h_α. As it is clear from the construction, these fields are just the anisotropies in the monodromy matrix, hence they can be assigned arbitrarily while – remarkably enough – retaining integrability. From earlier studies on the disordered model, it was concluded [80] that the effect of disorder is that of enhancing pairing correlations, while not modifying the crossover. In this section, we choose a Gaussian distribution for them, with mean $\overline{h} = 0$ and variance $\overline{h^2} = 1$.

Since the total spin S^z is conserved, a simplification comes from focusing on the subspace $S_z = 0$ (which exists for even N). Moreover, the integrability of the model allows us to go to spin numbers ($N = 50$ spins for single states and we will collect extensive statistics up to $N = 40$) which are sensibly higher with respect to those achievable by exact diagonalization and therefore allows to make some educated guesses on the thermodynamic limit of the system.

The picture that emerges from this analysis is that there is no many-body localization-delocalization phase transition in this model although the states appear de-localized on the computational basis for any finite g, the average single-spin observables are always localized.

3.2.1 Entanglement, average Hamming radius of an eigenstate and a local entropy

We will see that for all $g > 0$, $\log \mathcal{I} \propto N$, so an exponential number of sites of the hypercube of spin configurations is covered, although by the definition common in (single-particle) Anderson localization studies we would always find satisfied the limit

$$\lim_{N\to\infty} \frac{\mathcal{I}}{\binom{N}{N/2}} \to 0, \tag{3.47}$$

which flags instead a single-particle *localized* phase. In facts, the analysis of single-particle observables will confirm this scenario.

The amplitudes $\langle s_1, ..., s_N | E \rangle$ can be calculated as ratio of determinants of $(N/2) \times (N/2)$ matrices (therefore in time $\sim N^3$) once the roots w_j are known. However the number of terms in the sum is exponential in N so the calculation of $\mathcal{I}$ requires an exponential number of terms[2] and we are limited again to twenty spins or so.

[2]We have looked for a shortcut to evaluate this sum but to our knowledge integrability does not help us here.

We found two ways around this difficulty, they are complementary and we checked one against the other for consistency. First, we devised a Montecarlo algorithm for the evaluation of $\mathcal{I}$. Define the probabilities $p_a = |\langle a|E\rangle|^2$ where $a \in \mathcal{C}$ stands for one of the $\binom{N}{N/2}$ allowed configurations of spins which constitute the configuration space $\mathcal{C}$. We perform a random walk with the probabilities p_a's, namely start from a random configuration a and we try to move to a random one of the $(N/2)^2$ neighboring states, say b, by accepting the move with probability $\min(1, p_b/p_a)$. This involves only one computation of p_b, which takes time $\sim N^3$. The random walk proceeds in this way, generating a history of configurations a for which we can take the average over Montecarlo time of p_a. The inverse of this value gives $\mathcal{I}$.

The intensive quantity is $\log \mathcal{I}/N$, which can then be averaged over differen states and realizations. We observe that for all $g = O(1)$ the value of $\ln \mathcal{I} \propto N$, testifying then that each state occupies an exponential number of states in the configuration space.

The second method is to find another quantity which can be computed in polynomial time and to link it to $\mathcal{I}$. Since the average values $\langle E| s_\alpha^z |E\rangle$ can be expressed again in terms of determinants they can be calculated in $O\left(N^3\right)$ time: therefore one is led to consider a *micro-canonical* version the Edwards-Anderson order parameter associated to a single eigenstate

$$q(E) = \frac{4}{N} \sum_{\alpha=1}^{N} \langle E| s_\alpha^z |E\rangle^2 , \tag{3.48}$$

with this normalization $q \in [0, 1]$. The average over eigenstates is

$$q = \frac{1}{2^N} \sum_E q(E). \tag{3.49}$$

Following [70] we start with a slightly magnetized spin α in an infinite temperature state:

$$\rho_0 = (\mathbf{1} + \epsilon s_\alpha^z)/2^N \tag{3.50}$$

with magnetization $\langle s_\alpha^z \rangle_0 = \mathrm{Tr}\left[\rho_0 s_\alpha^z\right] = \epsilon/4$ (as $s_z^2 = 1/4$). The same magnetization at large time t in the diagonal approximation reads

$$\langle s_\alpha^z \rangle_\infty = \lim_{t\to\infty} \mathrm{Tr}\left[e^{-iHt} \rho_0 e^{iHt} s_\alpha^z\right] = \frac{\epsilon}{2^N} \sum_E \langle E| s_\alpha^z |E\rangle^2 . \tag{3.51}$$

Therefore, averaging over α we obtain the equality with eq. (3.49):

$$q = \frac{1}{N} \sum_\alpha \frac{\langle s_\alpha^z \rangle_\infty}{\langle s_\alpha^z \rangle_0}, \tag{3.52}$$

namely the previously defined EA order parameter is the average survival fraction of the initial magnetization after very long times.

We notice two more things:[81] one, that q is related to the average purity of the state (here we use the total $S^z = 0$):

$$q = \frac{2}{N} \sum_\alpha \mathrm{Tr}\left[\rho_\alpha^2\right] - 1 \tag{3.53}$$

and two, that q is related to the average Hamming distance of the points in configuration space when sampled with the probability distribution p_a:

$$d_{a,b} = \sum_{\alpha=1}^{N} \frac{1 - 4\langle a| s_\alpha^z |a\rangle \langle b| s_\alpha^z |b\rangle}{2}, \tag{3.54}$$

and multiplying by $p_a,\ p_b$ and summing over a, b we find:

$$L \equiv \langle d \rangle = \frac{N}{2}(1-q). \tag{3.55}$$

So q is computationally easy and it captures both some geometric properties of the covering of the configuration space by an eigenstate and the long-time correlation function for s^z. We averaged q over the spectrum (sample over typical states) and then over realizations (the number of which depends on the size of the system but it will never be less than 100).

We found this average $\overline{q}$ as a function of g for $g \in [0, 40]$ and $N = 16, ..., 38$ and studied the pointwise finite-size scaling (in the form $q_N(g) = q(g) + c_1(g)/N + c_3(g)/N^3$) to obtain the thermodynamic limit of q (see Figure 3.2). We fit the data using a ratio of polynomials with the condition that $q(0) = 1$ and we found that averaging over the state and the realization of disorder

$$\overline{q} = \frac{1 + 3 \times 10^{-8} g}{1 + 1.003g + 0.009g^2} \simeq \frac{1}{1+g} \tag{3.56}$$

works in the whole range of data to an error of at most 0.5%. We therefore conjecture this to be the correct functional form of the EA order parameter at infinite temperature.

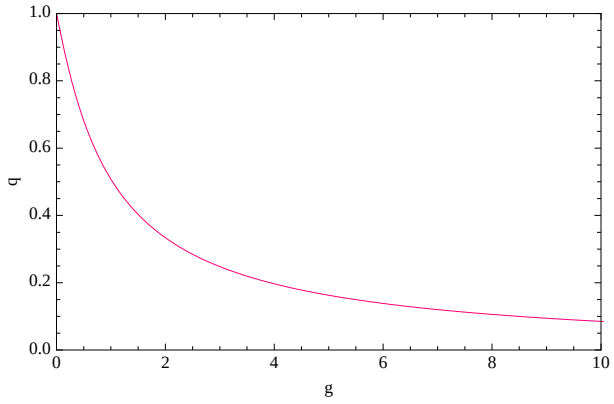


Figure 3.2: The pointwise extrapolation of the function q as a function of g. The fit $q = 1/(1+g)$ is not distinguishable from the data.

We can now go back to the relationship between the IPR and q, better expressed as a relation between $\log \mathcal{I}$ and L. We notice a one-to-one correspondence between average values these two quantities already at the level of second-order perturbation theory in g starting from a given state with $N/2$ spins up $\mathcal{S}_\uparrow$ and $N/2$ spins down $\mathcal{S}_\downarrow$:

$$\mathcal{I} = 1 + \frac{2g^2}{N^2} A + o(g^2) \tag{3.57}$$

where we defined a sum over pairs of up and down spins of the given state:

$$A = \sum_{\alpha \in \mathcal{S}_\uparrow, \beta \in \mathcal{S}_\downarrow} \frac{1}{(h_\alpha - h_\beta)^2} \tag{3.58}$$

Since A is dominated by small denominators, it will be typically $A = O(N^4)$ and therefore from the expression for IPR we see the perturbative regime is valid for $g \ll 1/N$. With an analogous computation we get:

$$L = \frac{4g^2}{N^2} A + o(g^2) \tag{3.59}$$

Eliminating g between the two relations and using (3.55) one gets, independently of the state and of the quenched randomness (therefore the relation holds also on average):

$$\log \mathcal{I} \simeq \frac{L}{2}. \tag{3.60}$$

So the relation is linear for small g. To see how this relation is modified at higher values of g we have again to resort to numerics. From the data it is clear that a strict relation exists between $\overline{\log \mathcal{I}}$ and $\overline{L}$ as one can see in Figure 3.3

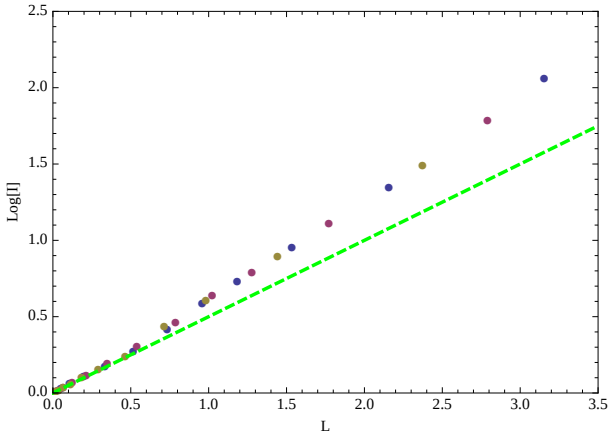


Figure 3.3: $\log \mathcal{I}$ as a function of the average distance L. The points are (blue, pink, yellow) $N = 28, 30, 32$ averaged over 100 realizations: the dashed straight line is the second order perturbation theory approximation Eq. (3.60).

By using the previous Montecarlo calculation for $\mathcal{I}$ we can plot $\log \mathcal{I}$ vs. L, showing that the relation is almost linear. The degree of non-linearity is measured by the ratio

$$s = \frac{\log \mathcal{I}}{2L} \tag{3.61}$$

which can be interpreted as a local entropy[82]. In fact, $2L = N(1-q)$ can be interpreted as the number of free spins (whose value of s^z is close to 0) while $\mathcal{I}$ is the number of configurations. If we want $2L$ spins to be responsible to $\mathcal{I}$ states then each of these spins should account for a degeneracy of e^s, from which the interpretation as an entropy density.

The distribution of L over states and realizations becomes more and more peaked as N grows, since we observe the variance $\delta L^2 \propto N$. The same occurs to $\log \mathcal{I}$, whose variance goes $\propto N$ in the region of g considered. Therefore the average value of s becomes typical in the large-N limit.

In the curves of Figure 3.4 the entropy s grows from the value of $1/4 = 0.250$ predicted by perturbation theory to an asymptotic value of $s = 0.383 \pm 0.003$.[3] This value is not what one would expect from a uniform superposition over $\binom{N}{N/2}$ states, since in that case $L = N/2$, $\log \mathcal{I} \simeq N \log 2$ and the familiar value $s = \log 2 = 0.693$ is roughly *twice* as much as we expect. This leads us to think that the most probable structure of the delocalized state at increasing g still retains a pair structure. We can build a toy model of delocalization in

[3] In the figure we show also a rational function best fit $s(g) = 0.403\ \frac{0.452+g}{0.656+g}$, which has however an error of 5% in the asymptotic value $s(\infty) = 0.403$ instead of 0.383, the value obtained by averaging on many more realizations and including smaller N in the fit.

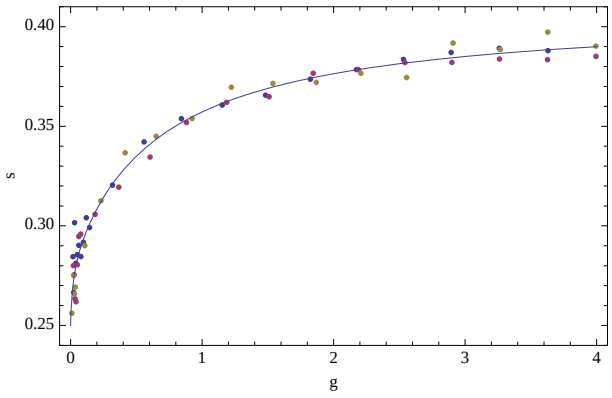


Figure 3.4: Local entropy as a function of g. The points are $N = 28, 32, 36$ (blue, pink, yellow) all averaged over 100 realizations. The fit is a (1,1)-Pade' approximation conditioned to $s(0) = 1/4$.

the typical eigenstates, by assuming that Nq spins are localized on their $g = 0$ values and that the remaining $N(1-q)$ spins are instead divided into couples, where couples are formed between almost resonating spins of opposite orientation. The couples are in one of the random valence bond states $|\uparrow\downarrow\rangle \pm |\downarrow\uparrow\rangle$ which are indeed the two $S_z = 0$ eigenstates of the two- body hamiltonian

$$H_2 = -\frac{g}{N}(s_1^+ s_2^- + s_2^+ s_1^-) - h(s_1^z + s_2^z) \tag{3.62}$$

where we have assumed that $h_1 \simeq h_2 = h$. This predicts that

$$\mathcal{I} \sim 2^{N(1-q)/2} = 2^L \tag{3.63}$$

and so that we should have a constant entropy $s = \log(2)/2 = 0.347$, slightly smaller than the observed value at large g and off by 40% at small g. The pair structure of a given eigenstate can be observed in Figure 3.5 where we plot the values of $m_n^2 \equiv \langle E| s_n^z |E\rangle^2$ for a given eigenstate $|E\rangle$, ordering the spins by increasing h_n (so that almost-resonant spins are nearest neighbours). We see a clear valence bond-like pair correlation in the values of the squared magnetization.

With the available data, we can discuss issues like the presence of multiple clusters in the same energy level $|E\rangle$. In fact, by randomly restarting the Montecarlo routine with the same p_a's if multiple clusters exist, we would expect to sample them according to their basin of attraction. Moreover we can rely on analytic results (such as those for q) to compare the Montecarlo averages with: clusterization and ergodicity-breaking would translate in a difference between these two results (as the random-walk would get stuck in a cluster and would not explore the whole configuration space). In the region where we can trust our numerics ($g \gtrsim 0.02$) Montecarlo averages converge to the analytic results, though a slowdown of the dynamics is observed (see below).

3.2.2 Dynamics of Montecarlo and other quantities

The Montecarlo routine which allows for importance sampling of the distribution p_a allows other measures of the geometry of the state. We can now study the similarities between the dynamics of importance sampling on p_a and that of random percolation on the hypercube, which has been proposed as a model of relaxation in a glassy system [82]. We will find that

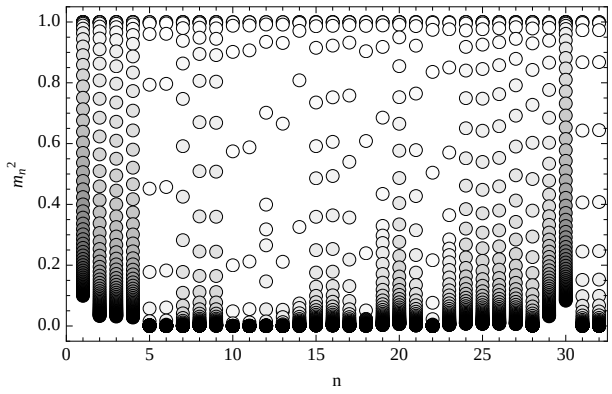


Figure 3.5: Squared magnetizations for increasing g (from white to black), where the spins are ordered by increasing magnetic field h_n. A consistent number of valence bond pairing is observed as a good fraction of neighboring spins (e.g. $n = 8, 9$, $n = 15, 16$ and $n = 31, 32$) have the same speed.

in both cases, a stretched exponential is the best fit and that the exponent depends on the coupling constant g. This, we believe, is a remarkable similarity.

An important quantity in this sense is the time dependence of the average distance from the starting point. Consider the Hamming distance $H(t)$ (t is Montecarlo time) from the starting point $H(t) = |a(t) - a(0)|$. For $t \gg 1$, $H(t)$ is fit quite accurately by a stretched exponential ansatz of the form:

$$H(t) = L\left(1 - e^{-\left(\frac{t}{\tau}\right)^{\beta}}\right), \tag{3.64}$$

where L is the average distance introduced before and β is a new characteristic exponent. Let us consider the behaviour of the exponent β with respect to g, as plotted in Fig. 3.6. Even if the results become quite noisy for small g, we can still see that for small values of g, β stays close to 1 (with some noise) as g increases, β decreases, although quite slowly. Instead for the time-scale τ we find, apart from the monotonic decrease with g, which is to be expected on general grounds, that for $g \gtrsim 1$ $\tau \propto N^{3/2}$, which we propose without explanation.

The small time behaviour of $H(t)$ can be used to obtain some information about the local structure of the state. In particular we can set

$$k \equiv \frac{H(1)}{2} = \frac{4}{N^2} \sum_{\langle a,b \rangle} \min(p_a, p_b) \tag{3.65}$$

where the last equality follows from the Montecarlo rate and the sum is over nearest-neighbour states. This quantity can be considered as a measure of the local connectivity, that is, the average fraction of active links.

From Fig. 3.7, we may deduce two things: one is that the connectivity stays well below 1 even for large g, confirming, as we claimed before, that the typical state is never uniformly spread over the hypercube; the second is that the connectivity scales with N as N^{-1} for small g and with $N^{-1/2}$ for large g (a fit $k = A/N^{\alpha}$ shows a continuously decreasing α from 1 to 1/2). Since the number of diverging roots is proportional to the total spin of the eigenstate at infinite g, and since the more roots diverge the more equally distributed the terms of each

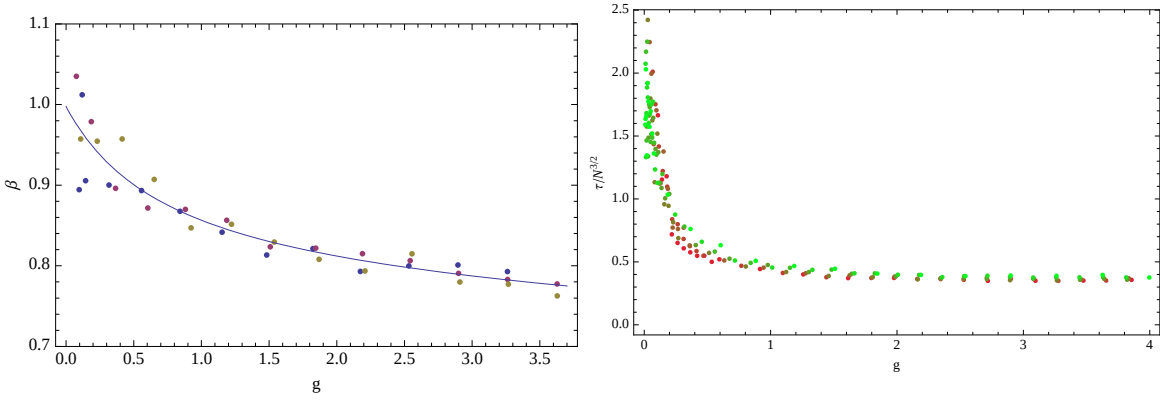


Figure 3.6: *Left.* The stretched exponential exponent β data as a function of g for $N = 28, 32, 36$ (blue, pink, yellow) together with a fit of the form $(1 + a_1 g)/(b_0 + b_1 g + b_2 g^2)$. *Right.* The timescale τ as a function of g for $N = 18, 20, ..., 34$ (red to green). The scaling $\tau \propto N^{3/2}$ is evidently good, in particular in the region $g > 1$.

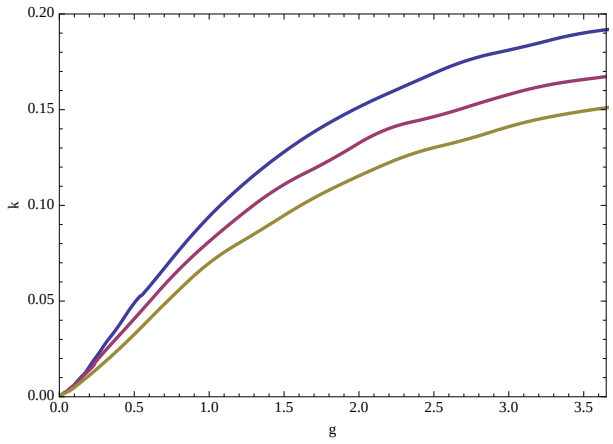


Figure 3.7: Local connectivity as a function of g. Different lines corresponds to $N = 18, 24, 30$ (blue, pink, yellow)

creation operator B are, we can argue the state will be better spread for larger total spin S. As for large N, the total spin of a typical state will increase only as $\sqrt{N}$ this explains the depletion of the local connectivity.

3.2.3 Setup of an exponential IPR

From perturbation theory (and from the Bethe-lattice approximation result [79]) one could argue that, if a phase transition occurs, it is at $g \sim 1/N$ (with or without the extra $1/\log N$, which would however not be noticed for our moderately large N). But does a phase transition in the geometric properties of the eigenstate occur?

First we analyze the quantity for which we have more extensive statistics (because of his polynomial complexity), L. A phase transition in L would mean that, set $\gamma = gN$, there exists a $\gamma_c > 0$ such that for $\gamma < \gamma_c$, $L/N \to 0$ and for $\gamma > \gamma_c$, $L/N \sim (\gamma - \gamma_c)^\delta$ where δ is a critical exponent. We have analized our data for $g > 0.01$ and $g < 0.2$ and we can conclude that this is not the behaviour observed. The behaviour is more consistent with $\gamma_c = 0$, $\delta = 1$ or with a crossover, in which the limit L/N when $N \to \infty$ is a smooth function of g which vanishes at $g = 0$. The matching with the part of the curve at finite g is smooth and the limiting behaviour is as described before.

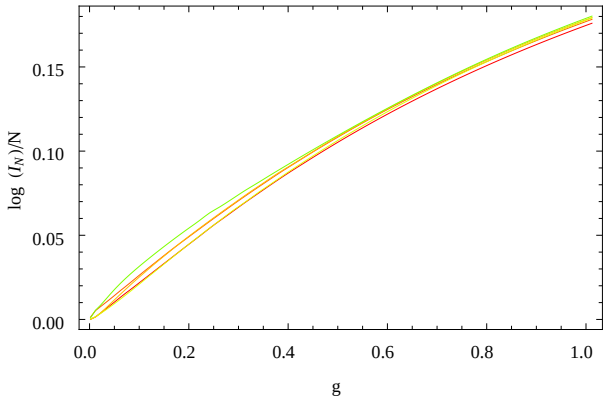


Figure 3.8: The logarithm of the inverse participation ratio divided by the size for N between 8 and 18, computed exactly and averaged over states and realizations. This graph shows no hint of a phase transition at $g \lesssim 1/N$.

There is the possibility however that although $L \sim N$ always, we have two phases: $\log \mathcal{I} \sim 1$ and $\log \mathcal{I} \sim N$ between which a transition occurs. This could happen if an eigenstate spread along one (or a few) directions without covering an exponential number of spin configurations. We have excluded this by both direct analysis of $\log \mathcal{I}/N$ data and by the observation that the relation between $\log \mathcal{I}$ and L remains valid all the way to small g (small here means $g \lesssim 1/N$). As $\log \mathcal{I}/N$ becomes soon independent of N without any scaling of g needed (see Fig. 3.8) we are led to conclude that no phase transition occurs as the system occupies an exponential number of sites of the computational basis for any $g > 0$.

3.2.4 Breaking of integrability

By considering the Richardson model essentially as a hopping process on the hypercube with random site energies given by the unperturbed energies we have found that the eigenstates are

always covering an exponential number of spin configurations but nonetheless $q > 0$ meaning thermalization is not achieved. From this point of view it is not clear which role, if any, integrability plays.

On the other hand, one would infer that the integrability of the model must play and essential role beyond providing the methods used for its solution. In particular, if the integrals of motion are too much "local", integrability can have the effect of freezing the expectation values of local quantities. To support such a claim we investigated a very similar non-integrable model, in which the hopping coefficients are not uniformly equal to g as in (1.86) but instead $N(N-1)/2$ random variables $g_{\alpha,\beta} = g(1+\epsilon\eta_{\alpha,\beta})$ where $\eta_{\alpha,\beta} = \eta_{\beta,\alpha} = \pm 1$ with probability $1/2$. Randomness in the fields is retained. The hamiltonian is

$$H = -\frac{g}{N}\sum_{\alpha,\beta} g(1+\epsilon\eta_{\alpha,\beta}) s_\alpha^+ s_\beta^- - \sum_\alpha h_\alpha s_\alpha^z. \tag{3.66}$$

We observe a decrease of the value of q (averaged both over E, η and h) as expected, in particular for sufficiently large g we are confident to say that $q \to 0$ for $N \to \infty$ and the system becomes ergodic.

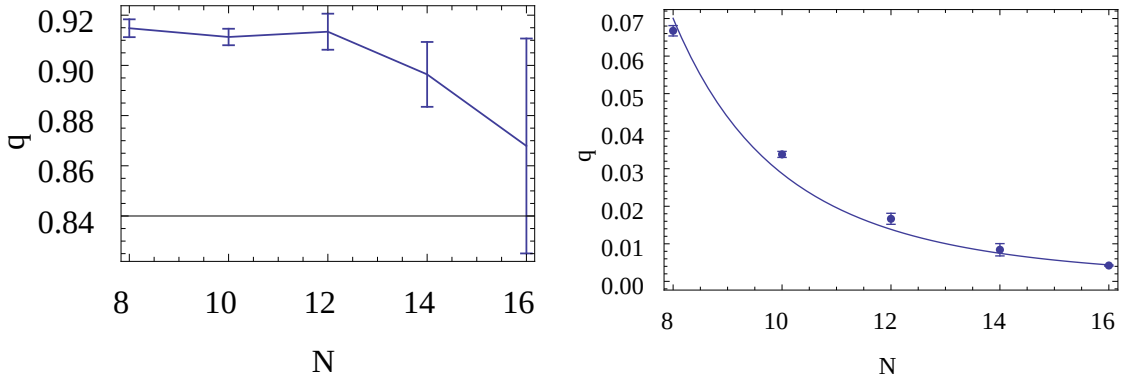


Figure 3.9: Averaged microcanonical q for the non-integrable hamiltonian eq. (3.66) for *up:* $g = 0.1,\ \epsilon = 0.4$ and *down*: $g = 4.1, \epsilon = 0.4$. The data in the lower panel are fit by a power law $aN^{-\gamma}$ with $a = 290$ and $\gamma \simeq 4$. The exponent γ seems to be g-dependent.

For small g however the situation is not so clear. The limit $N \to \infty$ could actually be zero or not, what is clear is that the N-dependence is not settled (compare the upper and lower panel of Figure 3.9) for $N = 16$, the largest system size that we can attain. This leads to two competing scenarios: in the first we have ergodicity as soon as $\epsilon > 0$; in the second, one could identify a finite $g_c(\epsilon)$ such that for $g < g_c$, $q > 0$ and for $g > g_c$ we have $q = 0$ in the thermodynamic limit. The latter would have a many-body localization transition at the said g_c. Much more extensive numerical work is needed to decide between these two scenarios. We leave the resolution of this issue for the future.

3.2.5 Conclusions and some directions for further work

We have performed a numerical study of typical states of the Richardson model with quenched disorder (an example of Gaudin magnet and an integrable system). We have found no evidence of a full delocalization phase transition although typical eigenstates occupy an exponential

number of states in the basis of s_i^z's for any $g > 0$. This better clarifies the "delocalized" scenario that has been introduced in previous works.

We have devised a method to calculate the IPR without summing over exponentially many states and studying its connections with a microcanonical version of the Edwards-Anderson order parameter, which measures the fraction of surviving magnetization at infinite temperature and for long times. Of this order parameter, we have conjectured the thermodynamic limit at infinite temperature as $q = 1/(1+g)$. We were unable to obtain the temperature dependence of this quantity, as sampling from the Boltzmann distribution is not straightforward within our framework.

For what concerns the absence of a many-body localization *phase transition* we can point out two peculiarities of our system as responsible for its absence. One is integrability and the other is the infinite range of the hamiltonian. We have therefore studied small-size systems (up to $N = 16$ spins) with an extra integrability breaking term of size ϵ. We observe a sharp reduction of q, which in some range of parameters could lead to think to a phase transition where $q = 0$ for $g > g_c(\epsilon)$. However it is possible that in the complementary region ($g < g_c(\epsilon)$) the decrease with N starts from a value of N impossible to reach with our limited numerics so we are unable to see that $q = 0$ for all g as soon as $\epsilon > 0$. Unfortunately this dichotomy is unlikely to be settled with the currently accessible values of N.

We point out that the Richardson model is one of a family of integrable spin systems (generalized Gaudin's magnets, see [83]) which can be studied with minor modifications of the methods introduced in this paper.

Part II

Integrability in field theory

Chapter 4

A review of integrable field theories in condensed matter

A system of particles moving in one dimension has no single-particle, but only *collective* excitations. This allows to treat the low-energy part of the spectrum of a wide class of problems, characterized by the feature of not showing gap in the thermodynamic limit, by the use of the same technique: it consists essentially in rewriting the Hamiltonian in terms of a "free" scalar bosonic field, a procedure that can be implemented effectively even for interacting systems [84].

As will be seen in the following, if one considers a one-dimensional system of fermions with a fairly general interaction, the density and the spin excitations can be decoupled. In particular, one can write a free effective action for the density part (Luttinger liquid), whereas for the spin part an additional term appears in the action, which may produce a *gapped* spectrum. This action is that of a particular integrable field theory: the sine-Gordon theory.

Moreover, even if one considers more general (bosonic, spin, ...) models or spinless fermions, there are many kinds of perturbations that can drive the action out of the Luttinger liquid point and produce a gap in the spectrum. We will recall, in particular, some of the (several) perturbations that produce an action of the sine-Gordon kind.

This chapter serves as a review of some aspects of quantum field theory in 1+1 dimension. The techniques that connect many problems of one-dimensional quantum physics with a fairly general low–energy action of quantum field theory are briefly reviewed in section 4.1, while we will focus onto the formalism of integrable field theories in section 4.2. The aim is to show that the original work on sine-Gordon presented in the next chapter has a general interest in one-dimensional physics.

4.1 Lattice systems in one dimension

Consider a system of fermions on a line. For the moment, following [18], we shall take them to be spinless and subject to the Tomonaga-Luttinger Hamiltonian:

$$H = -v_F \sum_k (k - k_F)\, c_k^\dagger c_k \tag{4.1}$$

where the Fermi velocity v_F and momentum k_F are parameters of the model.

Particle-hole excitations have a definite energy, which is independent of k_F and are therefore well-defined excitations that can be quantized. They are created by the transform of the density operator

$$: \rho(x)_r :=: \psi^\dagger(x)_r \psi_r(x) : \tag{4.2}$$

where the subscript r stands either for $R, +$ (right) or $L, -$ (left). Excitations with definite momentum are created by the operator

$$\rho_r^\dagger(q) = \begin{cases} \sum_k c_{r,k+q}^\dagger c_{r,k} & q \neq 0 \\ \sum_k : c_{r,k}^\dagger c_{r,k} := N_r & q = 0 \end{cases} \tag{4.3}$$

where the momentum q is measured with respect to k_F for right movers and to $-k_F$ for left movers. Being $\rho(x)$ real, we have $\rho(p)^\dagger = \rho(-p)$.

It follows from the expression above that

$$\rho_L^\dagger(p > 0)\,|0\rangle = 0\,, \qquad \rho_R^\dagger(p < 0)\,|0\rangle = 0 \tag{4.4}$$

which is to say that left movers with positive energy have negative momenta (less than the left Fermi edge), while right movers have positive momenta (bigger than the right Fermi edge).

One can show (see [18]) with periodic boundary conditions that

$$[\rho_r(q), \rho_s(p)] = -r \frac{q\,L}{2\pi} \delta_{r,s} \delta_{p,-q} \tag{4.5}$$

The ground state $|0\rangle$ of (4.1) is the state in which all the levels with negative energy are occupied and all the ones with positive energy are empty. According to this, let us now define bosonic creation and destruction operators as follows:

$$\begin{aligned} b_q^\dagger &= \sqrt{\frac{2\pi}{L|p|}} \rho^\dagger_{\mathrm{sgn}(q)}(q) \\ b_q &= \sqrt{\frac{2\pi}{L|p|}} \rho^\dagger_{\mathrm{sgn}(q)}(-q) \end{aligned} \tag{4.6}$$

A remarkable property of these operators is that also the operators associated to *fermionic* degrees of freedom can be written in terms of the (4.6). In particular, the Hamiltonian reads

$$H = v_F \left(\sum_{p \neq 0} |p| b_p^\dagger b_p + \frac{\pi}{L} N_r^2 \right) \tag{4.7}$$

Single-particle operators can be determined as well [84]: their expression is

$$\psi_r(x) = U_r e^{\sum_p e^{ipx}} \rho_r^\dagger(-p) \left(\frac{2\pi r}{pL} \right) \tag{4.8}$$

in which the operator U_r is a Klein factor, that suppresses one fermion of species r uniformly on the state where it acts and commutes with the boson operators.

It is convenient to introduce the fields

$$\phi(x) = -\frac{\pi}{L}(N_L + N_R)x - i\frac{\pi}{L} \sum_{p \neq 0} \frac{e^{-\alpha|p|/2}}{p} \left(\rho_R^\dagger(p) + \rho_L^\dagger(p) \right) \tag{4.9}$$

$$\theta(x) = \frac{\pi}{L}(N_L + N_R)x + i\frac{\pi}{L} \sum_{p \neq 0} \frac{e^{-\alpha|p|/2}}{p} \left(\rho_R^\dagger(p) - \rho_L^\dagger(p) \right) \tag{4.10}$$

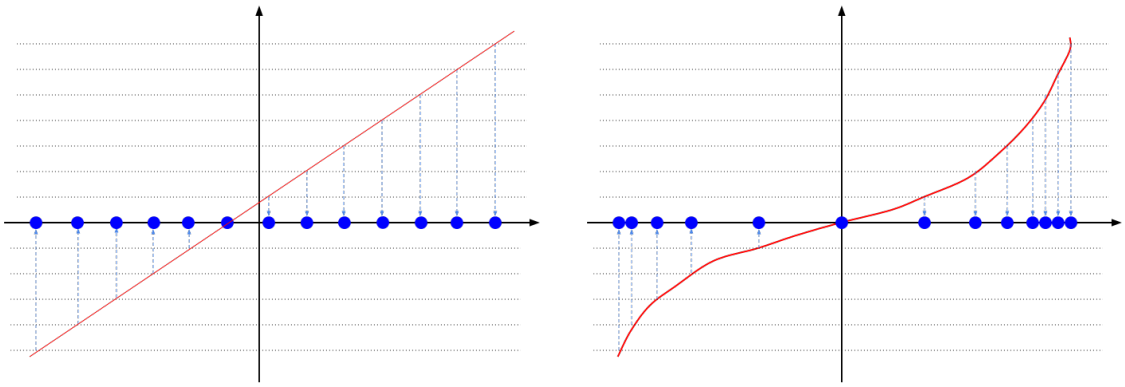


Figure 4.1: Counting function for a given number of particles in an homogeneous and a non homogeneous configuration: the slope of the function is proportional to the local density.

where a cutoff α has been introduced to regularize the infinite sum. Then (4.8) becomes

$$\psi_r(x) = U_r \lim_{\alpha \to 0} \frac{e^{ir(k_F - \pi p/L)x}}{\sqrt{2\pi\alpha}} e^{i(\theta(x) - r\phi(x))} \tag{4.11}$$

and the density of the two species of excitations can be reproduced as

$$\nabla\phi(x) = -\pi \left[\rho_R(x) + \rho_L(x)\right] \tag{4.12}$$

Formula (4.12) suggests a clear interpretation of the bosonic field ϕ as a *counting function*. To be precise, suppose that the position x of the $n-$th particle is identified by the condition

$$\phi(x_n) = 2\pi I_n \tag{4.13}$$

where I_n is some integer. Note that, since we are in one dimension, we can define an ordering for the particles, so that the function ϕ is always monotonic. It follows that in the regions in which the increase is the steepest, it will cross many quantization points in a relatively short interval, signalling a high density; conversely, regions in which ϕ is almost flat correspond to a low concentration of particles.

By rewriting the expressions above in terms of the bosons as from (4.6), one can show that in the infinite-volume limit

$$[\phi(x), \theta(y)] = i\frac{\pi}{2}\,\mathrm{sgn}(y - x) \tag{4.14}$$

hence the field

$$\Pi(x) = \frac{1}{\pi}\nabla\theta(x) \tag{4.15}$$

is canonically conjugated to the ϕ field

$$[\phi(x), \Pi(y)] = i\delta(y - x) \tag{4.16}$$

and the action becomes

$$H = \frac{v_F}{2\pi} \int dx \left[(\pi\Pi(x))^2 + (\nabla\phi(x))^2\right] \tag{4.17}$$

Note that the relation $\pi\Pi = \partial_t\phi/v_F - \pi(N_R - N_L)/L$ holds. The action associated to this Hamiltonian is the free Gaussian action

$$S_0 = \frac{1}{2\pi}\int d\tau dx \left[(\partial_\tau\phi)^2 - (\nabla\phi)^2\right] \tag{4.18}$$

after the change of variable $\tau \to v_F\tau$.

Interactions

It is now possible to consider interactions among fermions, in the rather general form

$$H_I = \int dxdy V(x-y)\rho(x)\rho(y) \tag{4.19}$$

In this situation, we cannot expect the dispersion relation to be linear as in the Tomonaga-Luttinger Hamiltonian. Nevertheless, if one considers low temperatures, in such a way that the excitations involve only a small region of the whole spectrum and that the curvature of the spectrum is not sensed, it is reasonable to linearize the dispersion relation in the vicinity of the two Fermi edges.

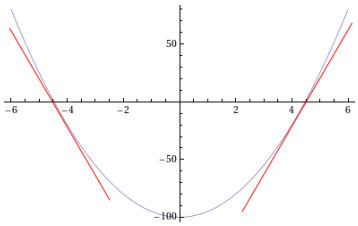


Figure 4.2: Linearization of the dispersion relation around the Fermi edges

The fermionic field then becomes

$$\psi(x) \simeq \frac{1}{\sqrt{L}}\left[\sum_{k\sim -k_F} e^{ikx}c_k + \sum_{k\sim k_F} e^{ikx}c_k\right] \tag{4.20}$$

and the density-density interaction can be rewritten in terms of left and right fields.

In a low energy description, interaction processes involve scattering of quasiparticles only in proximity of the Fermi edges and can be grouped in three categories [85]. They correspond to the terms that appear when using (4.20) to rewrite (4.19). The corresponding effective strengths are g_1, g_2 and g_4. In the first kind of process, a particle is scattered from the "left" to the "right" side of the Fermi sea, thus gaining a momentum $q \sim 2k_F$, while another one loses the same momentum and goes from around the "right" to around the "left" edge. The "g_2" process, instead, involves only the transfer of a small amount of momentum $q \ll k_F$ from one particle to another on opposite sides of the Fermi sea. Note that, in the absence of any internal degree of freedom, such as a spin, it will be undistinguishable from the first. The g_4 scattering, instead, is the momentum exchange among particles on the same side of the Fermi sea.

Approximating the two-body potential by a delta function one obtains momentum independent couplings, and the corresponding terms in the action are given by

$$g_4(\psi_R^\dagger\psi_R\psi_R^\dagger\psi_R + \psi_R^\dagger\psi_R\psi_R^\dagger\psi_R) = \frac{g_4}{(2\pi)^2}\left[(\nabla\phi)^2 + (\nabla\theta)^2\right] \tag{4.21}$$

$$g_2\psi_R^\dagger\psi_R\psi_L^\dagger\psi_L = \frac{g_2}{(2\pi)^2}\left[(\nabla\phi)^2 - (\nabla\theta)^2\right] \tag{4.22}$$

The g_1 processes, being the particles not provided with an internal degree of freedom, are not distinguishable from those of the g_2 type. The effect of the g_4 term is that of renormalizing the velocity of the excitation away from the Fermi velocity to

$$u = (v_F + \frac{g_4}{2\pi}) \tag{4.23}$$

Moreover, with respect to the the free bosonic action, the term g_2 represents a marginal perturbation, which can be absorbed at the Lagrangian level in a redefinition of the field. A new parameter $K \simeq 1 - \frac{g_2}{2\pi v_F}$ can be defined.

All the terms obtained are expressible by means the scalar fields (4.9), so that the final form only contains derivatives of θ and of ϕ in the *quadratic* effective Hamiltonian

$$H = \frac{u}{2\pi} \int dx \left[K \left(\pi \Pi(x)\right)^2 + \frac{1}{K} \left(\nabla \phi(x)\right)^2 \right] \tag{4.24}$$

where u sets the energy scale and K is a dimensionless parameter. Both are model-dependent and contain all the information about interactions of the original system. If these are weak, u and K can be determined perturbatively, while in the general case one needs to rely on the computation of thermodynamic quantities by other means, such as algebraic Bethe ansatz, to obtain them.

The corresponding action is given by:

$$S = -\int_0^\beta d\tau \int dx \left[\frac{1}{\pi} \partial_x \theta \partial_\tau \phi - \frac{1}{2\pi} \left(uK (\partial_x \theta)^2 + \frac{u}{K} (\partial_x \phi)^2 \right) \right] \tag{4.25}$$

but is more conveniently written in momentum space as:

$$S = \frac{1}{2\beta L} \sum_b k \left(\theta^*_{\mathbf{q}}, \phi^*_{\mathbf{q}} \right) \frac{1}{\pi} \begin{pmatrix} q^2 uK & ik\omega_n \\ iq\omega_n & q^2 u/K \end{pmatrix} \begin{pmatrix} \theta_{\mathbf{q}} \\ \phi_{\mathbf{q}} \end{pmatrix} \tag{4.26}$$

with the notation $\mathbf{q} = (\omega_n, q)$

If one is only interested in expectation values of composite operators of the field ϕ, such as, e.g., the magnetization correlation function in the Heisenberg chain, then the field θ can be integrated out by completing the corresponding square, which yields:

$$\begin{aligned} e^{-S} &= \exp \Big\{ \frac{1}{L\beta} \sum_{\mathbf{q}} \big[-\frac{\omega_n^2}{2\pi u K} \phi(\mathbf{q}) \phi(-\mathbf{q}) \\ &\quad - \frac{uKq^2}{2\pi} \left(\theta(\mathbf{q}) + \frac{i\omega_n}{uKq} \phi(\mathbf{q}) \right) \left(\theta(-\mathbf{q}) + \frac{i\omega_n}{uKq} \phi(-\mathbf{q}) \right) - \frac{uq^2}{2\pi K} \phi(\mathbf{q}) \phi(-\mathbf{q}) \big] \Big\} \end{aligned} \tag{4.27}$$

Then one is left with a gaussian integral in the variable θ, which cancels with the partition function in the correlators. Then, it is possible to write an action with respect to a single scalar field only as:

$$S = \frac{1}{2\pi K} \int_0^L dx \int_0^\beta d\tau \left[\frac{1}{u} \left(\partial_\tau \phi \right)^2 + u \left(\partial_x \phi \right)^2 \right] \tag{4.28}$$

Another way of obtaining this action is by noting that at the Gaussian point, the momentum is (proportional to) the derivative $\partial_\tau \phi$. With either the above action or its analogue containing only the θ variable one can obtain [18] the correlation functions

$$\begin{aligned} \left\langle \left(\phi(r) - \phi(0) \right)^2 \right\rangle &= K F_1(r) \\ \left\langle \left(\theta(r) - \theta(0) \right)^2 \right\rangle &= \frac{1}{K} F_1(r) \end{aligned} \tag{4.29}$$

with

$$F_1(r) = \frac{1}{2}\log\frac{x^2 + (u|\tau| + \alpha)^2}{\alpha^2} \tag{4.30}$$

while the mixed ones can be extracted from the complete action (4.26), by Fourier transform

$$\langle\phi(r)\theta(0)\rangle = \frac{1}{2}F_2(r) \tag{4.31}$$

where

$$F_2(r) = -i\arg\left(u\tau + \alpha\,\mathrm{sgn}\,\tau + ix\right) \tag{4.32}$$

It is sufficient to apply these formulas for evaluating all expectation values. For a quadratic Hamiltonian, in facts, one has

$$\left\langle e^{i\sum_j (A_j\phi(r_j) + B_j\theta(r_j))}\right\rangle = e^{1/2\sum_{j<k}\left[(A_jA_kK + B_jB_kK^{-1})F_1(r_j - r_k) - (A_jB_k + A_kB_j)F_2(r_j - r_k)\right]} \tag{4.33}$$

from which general correlation functions can be evaluated.

A remarkable advantage of the Luttinger liquid approach to the study of one-dimensional problems is that most *interacting* systems of spinless fermions allow a description in terms of a "free" bosonic effective model, under an appropriate choice of the parameters. Note that the parameter u in (4.28) just sets the relative scale of space and time and is incorporated in the time variable with $\tau \to u\tau$.

The parameter that encodes the information about the large-scale behaviour of the interacting system is K; according to (4.21), one has $K < 1$ for repulsive interactions and $K < 1$ for attractive interactions, at least at the perturbative level. Most importantly, it sets the scaling dimension of the field: whether a perturbation is relevant or not with respect to the Luttinger point ultimately depends on its value.

Fermions with spin

When fermions carry a spin degree of freedom it is again possible to use the boson representation (4.9), but one needs to treat separately the spin-up and the spin-down particles, by introducing two sets of fields $\phi_\uparrow, \theta_\uparrow$ and $\phi_\downarrow, \theta_\downarrow$. It turns out to be more convenient to separate the charge and spin degrees of freedom by defining the corresponding densities as:

$$\rho(x) = \frac{1}{\sqrt{2}}\left[\rho_\uparrow(x) + \rho_\downarrow(x)\right]\ , \qquad \sigma(x) = \frac{1}{\sqrt{2}}\left[\rho_\uparrow(x) - \rho_\downarrow(x)\right] \tag{4.34}$$

From these one writes the bosonic fields

$$\phi_\rho(x) = \frac{1}{\sqrt{2}}\left[\phi_\uparrow(x) + \phi_\downarrow(x)\right]\ , \qquad \phi_\sigma(x) = \frac{1}{\sqrt{2}}\left[\phi_\uparrow(x) - \phi_\downarrow(x)\right] \tag{4.35}$$

$$\theta_\rho(x) = \frac{1}{\sqrt{2}}\left[\theta_\uparrow(x) + \theta_\downarrow(x)\right]\ , \qquad \theta_\sigma(x) = \frac{1}{\sqrt{2}}\left[\theta_\uparrow(x) - \theta_\downarrow(x)\right] \tag{4.36}$$

A general density-density interaction term appears to be

$$H_I = \sum_{\sigma,\sigma'=\uparrow,\downarrow}\int dxdy V(x-y)\rho_\sigma(x)\rho_{\sigma'}(y) \tag{4.37}$$

Also in this case one considers the excitations around the Fermi edges only and rewrites the density-density interaction in terms of the analogues of the linearized fermion field. Again,

we shall consider the contact approximation for the interaction in (4.37), and denote by the subscripts $||$ the strength of the interaction among electrons with parallel spin and by $\perp$ the one among electrons with opposite spin. By using the fields (4.35), we can write the interaction terms as:

$$\begin{aligned} H_4 &= \frac{1}{2}\sum_{r=LR,\sigma=\uparrow\downarrow}\int dx\left[g_{4\,||}\rho_{r,\sigma}\rho_{r,\sigma}+g_{4\,\perp}\rho_{r,\sigma}\rho_{r,-\sigma}\right] \\ &= \frac{g_{4\,||}+g_{4\perp}}{4\pi^2}\int dx\left[(\nabla\phi_\rho)^2+(\nabla\theta_\rho)^2\right]+\frac{g_{4\,||}-g_{4\perp}}{4\pi^2}\int dx\left[(\nabla\phi_\sigma)^2+(\nabla\theta_\sigma)^2\right] \end{aligned} \tag{4.38}$$

and

$$\begin{aligned} H_2 &= \sum_{\sigma=\uparrow\downarrow}\int dx\left[g_{2\,||}\rho_{R,\sigma}\rho_{L,\sigma}+g_{2\perp}\rho_{R,\sigma}\rho_{L,-\sigma}\right] \\ &= \frac{g_{2\,||}+g_{2\perp}}{4\pi^2}\int dx\left[(\nabla\phi_\rho)^2-(\nabla\theta_\rho)^2\right]+\frac{g_{2\,||}-g_{2\perp}}{4\pi^2}\int dx\left[(\nabla\phi_\sigma)^2-(\nabla\theta_\sigma)^2\right] \end{aligned} \tag{4.39}$$

Here an essential difference arises. In facts, when particles carry spin, the processes of the g_1 kind that correspond to the scattering of a right mover with given spin to the left Fermi edge and a left mover with *opposite* spin to the right Fermi edge are to be treated separately. It can be split into two contributions. The first one is:

$$g_{1\,||}\sum_\sigma\int dx\psi^\dagger_{L,\sigma}\psi^\dagger_{R,\sigma}\psi_{L,\sigma}\psi_{R,\sigma}=-g_{1\,||}\sum_\sigma\int dx\rho_{L,\sigma}\rho_{R,\sigma} \tag{4.40}$$

which is indeed equal to a g_2 process. However, the term corresponding to the second contribution is

$$g_{1\perp}\sum_\sigma\int dx\psi^\dagger_{L,\sigma}\psi^\dagger_{R,-\sigma}\psi_{L,-\sigma}\psi_{R,\sigma}=\int\frac{dx}{(2\pi\alpha)^2}\cos(2\sqrt{2}\phi_\sigma(x)) \tag{4.41}$$

Summing up, the Hamiltonian for the *spin* excitations is given by

$$H=\int\frac{dx}{2\pi}\left[u_\sigma K_\sigma\left(\pi\Pi_\sigma(x)\right)^2+\frac{u_\sigma}{K_\sigma}\left(\nabla\phi_\sigma(x)\right)^2\right]+2g_{1\perp}\int\frac{dx}{(2\pi\alpha)^2}\cos(2\sqrt{2}\phi_\sigma(x)) \tag{4.42}$$

This is a sine-Gordon type Hamiltonian. The effect of the cosine term is that of trying to lock the field ϕ_σ into one of the minima of the cosine, therefore competing with the kinetic part that allows the field fluctuate. The resulting ground state out of this competition, can be determined from a renormalization group analysis. In Lagrangian form:

$$S_\sigma=\frac{1}{2\pi\,K_\sigma}\int dx\int d\tau\left[u_\sigma\left(\partial_\tau\phi_\sigma\right)^2+\frac{1}{u_\sigma}\left(\nabla\phi_\sigma\right)^2\right]-y\int dx\int d\tau\cos(2\sqrt{2}\phi_\sigma) \tag{4.43}$$

with $y=g_{1\perp}/(2\pi\alpha)^2$.

The scaling dimension of the perturbing operator at the conformal point (4.25) can be extracted from the operator product expansion of the operator with the energy-momentum tensor (see, e.g. [86]). For the vertex operator

$$\mathcal{V}_\alpha =: e^{i\alpha\phi} : \tag{4.44}$$

it is given by:

$$y_\alpha = 2\,\alpha^2\,K/8 \tag{4.45}$$

which implies that the perturbing operator $:\cos(2\sqrt{2}\phi):$ is relevant for $K < 1$ and irrelevant for $K > 1$.

Moreover, the appearance of a gap in the charge sector is also possible, when a relevant operator perturbs the Luttinger point. This is associated with the so–called *umklapp* terms, of type

$$\psi^\dagger_{L,\uparrow}\psi^\dagger_{L,\downarrow}\psi_{R,\uparrow}\psi_{R,\downarrow} \tag{4.46}$$

which usually occur at commensurate filling. This general formalism has many applications in one dimension. In particular, the XXZ spin chain (1.54) with interaction along the z axis smaller (in module) than those along the other two axes can be mapped into a free bosonic action; when the interaction parameter along z is antiferromagnetic and its absolute value bigger than one, the spectrum becomes gapped and the sine-Gordon action describes the low energy physics of the chain (see [18], [87]). Also, the chain in the regime $0 < \gamma < \pi$ with total magnetization equal to zero (or when the number of "flipped" spin with respect to the reference state is a fraction of the length), it may develop a gap, depending on the parameter γ and on the magnetization. Moreover, the Hubbard model, paradigm of hopping conduction, can be rewritten in the continuum limit as the massive Thirring model and bosonized into the sine-Gordon model. This and other examples are collected in [88].

4.2 Integrable massive field theories

In the spirit of section 4.1, we now turn to the definition of a class of two-dimensional quantum field theories that emerges as perturbation [89] of conformal field theories.

$$\mathcal{S} = \mathcal{S}_{CFT} + \mu \int d^2x\, \Phi \tag{4.47}$$

where Φ is a relevant operator of the conformal theory.

Sufficient conditions, ensuring that an infinite set of commuting charges exists also outside the critical point, can be given for certain families of perturbing operators [90, 89]. When this requirement is met, the theory is called *integrable*: the particle content, the associated scattering amplitudes and the two-point functions can be computed exactly.

4.2.1 Factorizable S-matrices

Let us denote the conserved charges by $\mathcal{Q}_s$, labelled by the corresponding spin s, which characterizes the transformation under a Lorentz boost L_α. Given a boost that shifts the particles of a state as $L_\alpha\,|\theta_1,\ldots,\theta_n\rangle \to |\theta_1+\alpha,\ldots,\theta_n+\alpha\rangle$, then the corresponding transformation of the charge under the boost can be written as

$$U_{L_\alpha}\mathcal{Q}_s U^\dagger_{L_\alpha} = e^{s\alpha}\mathcal{Q}_s \tag{4.48}$$

The set of charges must include the light-cone momenta

$$p = p^0 + p^1\,, \qquad \bar{p} = p^0 - p^1 \tag{4.49}$$

with eigenvalues $\omega_1 = m\,e^{\theta}$ and $m\,e^{-\theta}$, respectively. Under a Lorentz boost $\theta \to \theta + \alpha$, they transform with spin ± 1, according to the previous definition. The energy-momentum operator, which is conserved after translational invariance in the two direction, is a linear combination of the two.

Physically relevant problems involve generally interactions among fields; nonetheless, having in mind a scattering problem, it is convenient to consider a basis of ingoing or outgoing *asymptotic* states, in which excitations are so far away one from the other that they do not interact. Specifically, a "particle" basis can be defined by the requirement that all charges act diagonally on it:

$$\mathcal{Q}_s \left|p_1, \ldots, p_M\right\rangle_{a_1,\ldots,a_M} = \sum_{j=1}^{M} \omega_s^{a_j}(p_j) \left|p_1, \ldots, p_M\right\rangle_{a_1,\ldots,a_M} \tag{4.50}$$

In particular, eigenvalues of all charges are additive in the particles.

Suppose to have a set of n particles with a species label a. In the remote past, particles are arranged with decreasing momenta

$$\left|p_1, \ldots, p_n\right\rangle^{in}_{a_1,\ldots,a_n} \qquad p_1 > p_2 > \ldots > p_n \tag{4.51}$$

One may think to such a state, as an heuristic guide, as having the particles in definite positions $x_1 < x_2 < \ldots < x_n$ as well. A scattering process has taken place, in the far future, particles are arranged with opposite ordering for coordinates and momenta

$$\left\langle q_1, \ldots, q_n\right|_{b_1,\ldots,b_n} \qquad q_1 < q_2 < \ldots < q_n \tag{4.52}$$

These sets of states serve as a *complete* basis for the "in" and "out" Fock spaces.

$$\sum_M \frac{1}{M!} \int dp_1 \ldots dp_M \left|p_1, \ldots, p_M\right\rangle_{a_1,\ldots,a_N} \cdot_{a_1,\ldots,a_N} \left\langle p_1, \ldots, p_M\right| = \mathbf{1} \tag{4.53}$$

Then, a *scattering matrix* can be defined by

$$\left|p_1, \ldots, p_M\right\rangle^{in}_{a_1,\ldots,a_N} = \sum_M \sum_{q_1,\ldots,q_M;b_1,\ldots,b_M} S^{b_1,\ldots,b_M}_{a_1,\ldots,a_N}(p_1, \ldots, p_N; q_1, \ldots, q_M) \left|q_1, \ldots, q_M\right\rangle^{out}_{b_1,\ldots,b_M} \tag{4.54}$$

where the set $\{a\}$ and $\{b\}$ label the particle kind or any other internal quantum number. The S-matrix can be defined as an operator by specifying its matrix elements

$$\begin{matrix} in/out \\ a_1,\ldots,a_N \end{matrix} \left\langle p_1, \ldots, p_M\right| \hat{S} \left|q_1, \ldots, q_M\right\rangle^{in/out}_{b_1,\ldots,b_M} \tag{4.55}$$

In other words, it gives the probability amplitude of finding the system in a given final state long after the collision, given the specified initial state long before the collision.

$$P_{a_1,\ldots,a_N \to b_1,\ldots,b_M} = \left| {}_{a_1,\ldots,a_N}\left\langle p_1, \ldots, p_M\right| \hat{S} \left|q_1, \ldots, q_M\right\rangle_{b_1,\ldots,b_M} \right|^2 \tag{4.56}$$

The presence of these charges and the fact that they transform differently under the Lorentz group implies [91] that, in any process, the scattering matrix (4.54) factorizes in the product of two-particle S-matrices. Moreover, the fact that there are an *infinite* number of charges implies that in any scattering process particle production cannot take place, i.e., that the final number of particles of given mass m_a after a scattering process is the same as the initial one, and that the final set of two-momenta is the same as the initial one [6].

In a particle description of a relativistic field theory, it is customary to parametrize the energy-momentum of a particle by using a *rapidity* θ as

$$p^0 = m \cosh\theta \quad , \qquad p^1 = m \sinh\theta \tag{4.57}$$

which ensures the on-shell condition

$$\left(p^0\right)^2 - \left(p^1\right)^2 = m^2 \tag{4.58}$$

By relativistic invariance, the two-particle scattering amplitude between a and b can only be a function of a Lorentz scalar. In a two-dimensional world, there exist only two such independent Mandelstam variables characterizing the process, namely s and t, which correspond to the processes

$$\begin{array}{ll} s & a+b \to c+d \\ t & a+\bar{d} \to c+\bar{b} \end{array}$$

in which the $\bar{k}$ denotes the antiparticle of kind k. It is customary to use the total energy in the center-of-mass frame:

$$s = (p_a + p_b)^\mu (p_a + p_b)^\nu \eta_{\mu\nu} = m_a^2 + m_b^2 + 2m_a m_b \cosh(\theta_a - \theta_b) \tag{4.59}$$

In the "crossed" channel, the relevant variable is instead the relative squared momentum

$$t = (p_a - p_d)^\mu (p_a - p_d)^\nu \eta_{\mu\nu} = m_a^2 + m_d^2 - 2m_a m_d \cosh(\theta_a - \theta_d) \tag{4.60}$$

From the parametrization (4.59), the two-particle S-matrix can be written as a function of the rapidity difference and is defined as

$$|\theta_a, \theta_b\rangle_{a,b}^{in} = \sum_{c,d} S_{a,b}^{c,d}(\theta_a - \theta_b) \, |\theta_b, \theta_a\rangle_{c,d}^{out} \tag{4.61}$$

This matrix has to meet the analytic structure of a general scattering matrix [92], with the major simplification that no particle production can take place. When formulated in terms of the rapidity variable, it is possible to identify a *physical strip* $0 < \Im\,\theta < \pi$, which is separated by a cut from a *second strip* $\pi < \Im\,\theta < 2\pi$. Because of the overall $2\pi i$-periodicity of the parametrization, these two strips determine the function over all the complex plane.

The Yang-Baxter equation is, in this context, a relation that constrains the two-particle S–matrix of the theory. It can be derived as a consequence of the factorization of the full scattering matrix, since the process involving three ingoing particles a, b, c with rapidities $\theta_a, \theta_b, \theta_c$ and three outgoing particles a', b', c' with permuted rapidities can be written as:

$$\begin{aligned} &{}_{a',b',c'}^{out}\langle \theta_{a'}, \theta_{b'}, \theta_{c'} | \theta_a, \theta_b, \theta_c \rangle_{a,b,c}^{in} = \\ &\qquad S_{a,b}^{e,d}(\theta_{1,2}) \, S_{d,c}^{a',f}(\theta_{1,3}) \, S_{d,f}^{b',c'}(\theta_{2,3}) = S_{b,c}^{d,e}(\theta_{2,3}) \, S_{a,e}^{f,c'}(\theta_{1,3}) \, S_{f,b}^{a',b'}(\theta_{1,2}) \end{aligned} \tag{4.62}$$

In other words, the S-matrix factorization is well-defined only if the Yang-Baxter equation is satisfied. This factorization can be argued by considering, instead of single-particle states, wavepackets centred around a position. Then different shifts of the centre of the packet can be induced by the application of the different charges, without changing the amplitude of the process, which only involves momenta.

The relation (4.61) and orthonormality of both *in*- and *out*-states, together with the fact that in an *integrable* field theory we cannot have intermediate states with a different number of particles, imply

$$\sum_{b_1,b_2} S_{a_1,a_2}^{b_1,b_2}(\theta) \left[S_{b_1,b_2}^{c_1,c_2}(\theta)\right]^* = \delta_{a_1}^{c_1} \delta_{a_2}^{c_2} \tag{4.63}$$

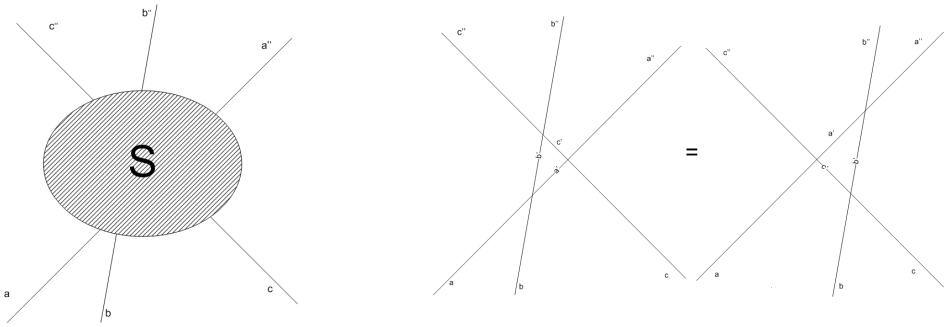


Figure 4.3: Yang-Baxter equation in the context of integrable massive quantum field theories: factorization of the three-particle scattering into a product of two-particle processes

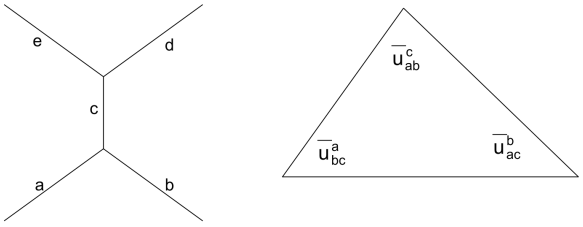


Figure 4.4: Left: graphical representation of poles in the S–matrix. Right: the mass triangle.

which is the *unitarity* of the two-particle S-matrix and is the analogue of (1.3).

It is essential that quantizing a theory in a scheme where space and time are rotated by $\pi/2$ gives the same scattering amplitude, which results in the requirement that the amplitude is the same when $\theta \to i\pi/2 - \theta$. Then a particle when the s and in the t channel are exchanged, is the same:

$$S^{c,d}_{a,b}(\theta) = C\; S^{c,b}_{a,d}(i\pi - \theta)\; C \tag{4.64}$$

with [93] the matrix C being a charge conjugation matrix that satisfies $C^t = C$ and $C^2 = \mathbf{1}$. This matrix acts on one particle space only and depends on the specific theory under examination. We have $C = i\sigma^2$ for the massive Thirring model and $C = \sigma^1$ for the soliton-antisoliton sector of the sine-Gordon S-matrix. this is called *crossing* invariance, and is formally equivalent to (1.6) apart from the overall scalar function ρ.

Poles of the scattering amplitude, if present, correspond to *bound states*, either in the direct or in the crossed channel. In the variable θ, they appear in the imaginary axis in the physical strip. The residue is related to the on-shell vertex $\Gamma^c_{a,b}$ of the incoming particles a, b and the bound state c:

$$S^{e,f}_{a,b}(\theta) \sim \frac{i\Gamma^c_{a,b}\Gamma^c_{e,f}}{\theta - iu^c_{a,b}} \tag{4.65}$$

By specifying the parametrization (4.59), it is possible to recognise the triangular relation

$$m_c^2 = m_a^2 + m_b^2 - 2m_a m_b \cos \bar{u^c_{ab}} \tag{4.66}$$

among the masses of the particles involved, where $\bar{u} = \pi - u$ (see Figure 4.4). Moreover, the charges are conserved through the fusion processes $a + b \to c$ and it follows that the set of

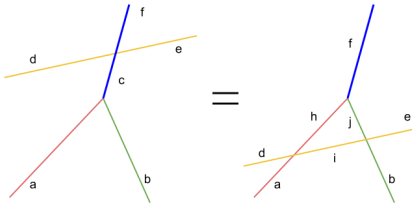


Figure 4.5: Graphical representation of the bootstrap equations

conserved charges of the theory, through the eigenvalues (4.50) for different spins, provide nontrivial constraints on the particle content of the theory.

The *bootstrap* approach to the problem of determining the scattering amplitudes assumes that the set of intermediate bound states and the set of asymptotic states are the same. This has the important consequence of allowing the determination of the scattering amplitude of a particle, say d with a particle c in the various channels, by only knowing the scattering matrices with the constituents a and d of c. This results in the *bootstrap equations*

$$\Gamma^c_{ab} S^{ef}_{dc}(\theta) = \Gamma^f_{hj} S^{ih}_{da}(\theta - i\overline{u^b_{a\bar{c}}}) S^{ej}_{ib}(\theta + i\overline{u^{\bar{a}}_{b\bar{c}}}) \tag{4.67}$$

This together with (4.62) and the constraints of unitarity and crossing, allow to determine both the particle content and the S-matrix iteratively [94].

4.2.2 Integrable deformation of free bosonic theories

According to section 4.1, one of the most interesting scaling models in one dimension is the Gaussian field theory, with the action:

$$S = \frac{1}{8\pi} \int d\tau \int^L dx \partial_\mu \phi(\tau, x) \partial^\mu \phi(\tau, x) \tag{4.68}$$

on a cylinder of circumference L. This action has conformal symmetry.

We require quasiperiodic boundary conditions for the field:

$$\phi(\tau, x + L) = \phi(\tau, x) + 2\pi m R \,, \qquad m \in \mathbf{Z} \tag{4.69}$$

The latter amounts to say that when the spatial coordinate travels across the support, the field winds m times around its compact target space, which is just a circle of radius R. It follows from the mode expansion of the field (see [86]) that the eigenvalues of the field momentum:

$$\pi_0 = \int_0^L dx \pi(\tau, x) \,, \quad \pi(\tau, x) = \frac{1}{4\pi} \nabla \phi(\tau, x) \tag{4.70}$$

are quantized as n/R, $n \in \mathbf{N}/2$, otherwise vertex operators are no longer well-defined. The topological charge:

$$Q = \frac{1}{R} \int_0^L \frac{dx}{2\pi} \partial_x \phi(\tau, x) = \frac{\phi(\tau, L) - \phi(\tau, 0)}{2\pi R} \tag{4.71}$$

has integer eigenvalues, given by the winding number m. It is also useful to define the fraction of topological charge in a given interval:

$$Q(x) = \frac{1}{R}\int_0^x \frac{dx'}{2\pi}\partial_{x'}\phi(\tau, x') = \frac{\phi(\tau, x) - \phi(\tau, 0)}{2\pi R} \tag{4.72}$$

where translational invariance implies that only the width of the integration interval is relevant.

It should be evident the relation of the compactified (4.68) with the Luttinger action (4.28). In facts, the velocity parameter u of the Luttinger model amounts to a redefinition of the time scale $\tau \to u\tau$ (the time direction is infinite). One then sees that the model is free for $K = 1$, which corresponds to a period of 2π on the target space of the free boson, i.e., to a compactification radius $R = 1$. On the other hand, a redefinition of the fields as

$$\phi \to R\phi \tag{4.73}$$

yields the condition (4.69) back in the "free" form, but with an explicit presence of the compactification radius in the action

$$S = \frac{R^2}{8\pi}\int d\tau \int_0^L dx \partial_\mu\phi(\tau, x)\partial^\mu\phi(\tau, x) \tag{4.74}$$

from which one obtains the relation

$$R^2 = \frac{4}{K} \tag{4.75}$$

so that, with respect to the action (4.74), the dimension of the vertex operator (4.44) is:

$$y_\alpha = \frac{\alpha^2}{R^2} \tag{4.76}$$

The action of the sine-Gordon field theory on a cylinder of radius L is usually found, in literature, written as:

$$\mathcal{S} = \int_{-\infty}^{\infty} d\tau \int_0^L dx \left[\frac{1}{2\beta^2}\partial_\nu\phi\partial^\nu\phi - \frac{\mu^2}{\beta^2}\cos(\phi)\right] \tag{4.77}$$

where μ and β are real parameters. With respect to the previous parametrizations, there holds:

$$\beta^2 = \frac{4\pi}{R^2} = \pi K \tag{4.78}$$

which implies that operators (4.44) have dimension

$$y_\alpha = \frac{\alpha^2\beta^2}{4\pi} \tag{4.79}$$

In other words, the cosine term in the action constitutes a relevant perturbation, provided

$$\beta < \sqrt{8\pi} \tag{4.80}$$

Terms containing multiples of the field in the cosine, such as the ones arising from the perturbations studied above, can be equally treated, by proper resizing of the field. Only the theory that contains one cosine term is integrable and will be considered in the following;

nevertheless, multiple perturbations can still be treated with the formalism of integrable field theory along the lines of [95, 96].

In infinite volume, the fundamental excitations are known to be the soliton, with mass $m = m(\mu^2)$ and unit topological charge, and the antisoliton, with equal mass and opposite charge. In particular, the mass gap is related to the mass parameter in the Lagrangian [97] by:

$$\mu = \frac{2\Gamma(B)}{\pi\Gamma(1-B)} \left(\frac{\sqrt{\pi}\Gamma\left(\frac{1}{2-2B}\right) m}{2\Gamma\left(\frac{\Delta}{2-2B}\right)} \right)^{2-2B} \quad , \qquad B = \frac{\beta^2}{8\pi} \tag{4.81}$$

A soliton and an antisoliton can bind together and form a breather with mass

$$m_{B_n} = 2m \sin \frac{n\, p\, \pi}{2} \tag{4.82}$$

and breather scattering can lead to the production of a higher mass breathers as well. To be specific, bound states are labelled by integers and their number depends on the value of β. By defining the parameter $p = \frac{\beta^2}{8\pi - \beta^2}$, it is possible to distinguish two regimes: a *repulsive* one, in which only the soliton and the antisoliton are present in the spectrum, and an *attractive* one, in which a number $\lfloor 1/p \rfloor$ of bound states are allowed.

The sine-Gordon S-matrix has a structure which derives from the topological charge of the elementary excitations and can be represented on the soliton-antisoliton sector, as:

$$\mathcal{S}_{SG}(\theta) = S(\theta) \begin{pmatrix} 1 & & & \\ & b(\theta) & c(\theta) & \\ & c(\theta) & b(\theta) & \\ & & & 1 \end{pmatrix} \tag{4.83}$$

where the overall scattering phase is:

$$S(\theta)^{++}_{++} = S(\theta)^{--}_{--} = S(\theta) \quad = \quad -\exp\left(\int \frac{dx}{x} e^{-2i\theta x} \frac{\sinh\left(1-p\right)\pi x}{\cosh \pi x \, \sinh \pi p x} \right) \tag{4.84}$$

and the soliton-antisoliton components are:

$$\begin{aligned} b(\theta) = S(\theta)^{+-}_{+-} = S(\theta)^{-+}_{-+} \quad &= \quad \frac{\sinh(i\frac{\theta}{p})}{\sinh(\frac{i\pi-\theta}{p})} S(\theta) \\ c(\theta) = S(\theta)^{-+}_{+-} = S(\theta)^{+-}_{-+} \quad &= \quad \frac{\sinh(i\pi/p)}{\sinh(\frac{i\pi-\theta}{p})} S(\theta) \end{aligned} \tag{4.85}$$

Please note that, apart from the overall phase, it has exactly the same form as (1.4), with the spin orientation degree of freedom replaced by the topological charge. Given the discussion of Chapter 1, and in particular the underlying algebraic constraints (4.64,4.63,4.62), this should not be surprising at all. Soliton-breather and breather-breather scattering can be determined through the bootstrap equations above and can be found, e.g., in [98, 93].

The massive Thirring field theory describes an interacting Dirac fermion in two dimensions

$$\mathcal{S} = \int_{-\infty}^{\infty} d\tau \int_0^L dx \left[\bar{\Psi}(i\gamma_\mu \partial^\mu + m)\Psi - \frac{g}{2} \bar{\Psi}\gamma_\mu \Psi \bar{\Psi}\gamma^\mu \Psi \right] \tag{4.86}$$

where the gamma matrices are

$$\gamma^0 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}, \qquad \gamma^5 = \gamma^0\gamma^1 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \tag{4.87}$$

and the spinor and the vector current are

$$\Psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}, \qquad J_\mu = \bar{\Psi}\gamma_\mu\Psi \tag{4.88}$$

The S–matrix of the Thirring model has been determined in [99] and the scattering of fermions was found to coincide with the one of sine-Gordon solitons, apart from an overall phase factor. In the attractive regime, bound states of fermions are seen to scattered with the amplitudes of the sine-Gordon breathers.

Both theories can be seen as relevant perturbations of a $c = 1$ conformal field theory. In the sine-Gordon model, the theory is a free bosonic theory and the perturbation is given by the cosine term in (4.77), while the massless Thirring model is a $c = 1$ theory of chiral free fermions, perturbed by a mass term. Correlation function of the perturbing fields $\bar{\Psi}\Psi$ and $:\cos\phi:$ are equal in the neutral sector [100], under the identification

$$\frac{4\pi}{\beta^2} = 1 + \frac{g}{\pi} \tag{4.89}$$

In facts, both models can be seen as perturbations by the same operator of two mutually nonlocal sectors of the operator content of the Gaussian theory [101].

At the operator level, it was shown in [102] that a fermionic operator satisfying the massive Thirring equation of motion can be constructed as a nonlocal functional of a bosonic field that satisfies the sine-Gordon equation of motion. The boson and the fermion, however, are not mutually local, hence they do not interpolate the same particle. In facts, the two theories are not equivalent [101], since they can be seen as perturbations of the, respectively, *bosonic* and *fermionic* $c = 1$ conformal field theories associated with the Gaussian fixed point.

Nevertheless, the charge associated to the lattice $U(1)$ symmetry can be identified and put in one-to-one correspondence in the two theories at the operator level. Then, both theories possess different sectors labelled by the "topological charge" (in SG), or the "fermion number" (in mT); in the common subspace with even topological charge, among which the vacuum [100], the two theories are indeed equivalent.

4.2.3 Correlation functions in integrable field theories

Solving a quantum field theory means to be able to compute all its multipoint functions exactly, the most relevant of which, being related to the response of the theory to the change of external parameters, are two-point functions. In the case of field theories with factorized scattering, an elegant and rather general approach for tackling the problem is provided by the form factor approach.

We start from a genera two-point function $\langle \mathcal{O}(x,t)\,\mathcal{O}(0,0)\rangle$. Let's suppose that we have, within the operator content of our theory, a field such that

$$\langle 0|\,\phi_I(x)\,|\theta_1,\ldots,\theta_n\rangle_{a_1,\ldots,a_n} \neq 0\,, \qquad n = 1,2,\ldots \tag{4.90}$$

this field is said to *interpolate* the particles. Let us now exploit the completeness of the set of asymptotic states to insert the decomposition of the identity in between of the operators:

$$\sum_{n=0}^{\infty} \frac{1}{n!} \int \frac{d\theta_1}{2\pi} \dots \frac{d\theta_n}{2\pi} |\theta_1 \dots \theta_n\rangle\langle\theta_n \dots \theta_1| \tag{4.91}$$

This defines [103, 104, 93] the form factor of the operator $\mathcal{O}$

$$\mathcal{F}^{\mathcal{O}}(\theta_1, \dots, \theta_n)_{a_1,\dots,a_n} = \langle 0| \mathcal{O} |\theta_1, \dots, \theta_n\rangle_{a_1,\dots,a_n} \tag{4.92}$$

The basic assumption is that $\mathcal{F}$ is a meromorphic function of its arguments, i.e., the particles rapidities $\theta_1, \dots, \theta_n$, within the strip $0 < \Im(\theta_i - \theta_j)$. Moreover, it is possible to further constrain the functional form by a set of equations derived from general principles of quantum field theory and from the structure of the two-particle S-matrix.

- Relativistic invariance

  $$F_n^{\mathcal{O}}(\theta_1 + \alpha, \dots \theta_n + \alpha) = e^{s\alpha} F_n^{\mathcal{O}}(\theta_1, \dots \theta_n) \tag{4.93}$$

  where s is the Lorentz spin, that characterizes the transformation of the field $\mathcal{O}$ under a boost.

- Watson theorem
  transfers the notion of particle scattering as in (4.61) to the form factor analytic structure, by defining the analytic continuation to states where rapidities are not ordered.

  $$\begin{aligned}\mathcal{F}^{\mathcal{O}}(\theta_1, \dots \theta_j, \theta_{j+1} \dots \theta_n)_{a_1,\dots,a_j,a_{j+1},\dots,a_n} = \\ \sum S^{b,c}_{a_j,a_{j+1}}(\theta_j - \theta_{j+1}) \mathcal{F}^{\mathcal{O}}(\theta_1, \dots \theta_{j+1}, \theta_j \dots \theta_n)_{a_1,\dots,b,c,\dots,a_n}\end{aligned} \tag{4.94}$$

- Locality
  An important concept in two-dimensional field theory is *mutual locality*. To be more precise, one considers the the relative displacement of two fields in terms of the variables $z = x + i\tau$ and $\bar{z} = x - i\tau$. Then, when performing a rotation $e^{i\alpha} z$, $e^{-i\alpha}\bar{z}$ with $\alpha = 0 \to 2\pi$, any correlator

  $$\langle \dots \phi_j(z, \bar{z}) \phi_k(0, 0) \dots \rangle \to e^{2\pi i \gamma_{jk}} \langle \dots \phi_j(z, \bar{z}) \phi_k(0, 0) \dots \rangle \tag{4.95}$$

  When not specified otherwise, the notion of *locality* of a field is intended to be referred to the Hamiltonian density $h(x)$, i.e

  $$\big[h(x), \phi(x')\big] = 0\,, \qquad x \neq x' \tag{4.96}$$

  The form factor inherits semilocality of the field $\mathcal{O}$ with respect to the field which interpolates the particle

  $$\mathcal{F}^{\mathcal{O}}(\theta_1 + 2\pi i, \dots\dots \theta_n)_{a_1,\dots,a_n} = e^{2\pi i \gamma_{\phi_a,\mathcal{O}}} \mathcal{F}^{\mathcal{O}}(\theta_2, \dots \theta_n, \theta_1)_{a_2,\dots,a_n,a_1} \tag{4.97}$$

- Crossing
  Let us consider the matrix element

  $$_{\bar{a}_1}\langle \theta_1| \mathcal{O} |\theta_2 \dots \theta_n\rangle_{a_2 \dots a_n} \tag{4.98}$$

The latter is interpreted as the interaction of a set of particles with labels from 2 to n with the operator $\mathcal{O}$, projected on an *out* state containing only one particle. If the rapidity and the particle kind on the left correspond to one on the right, this is interpreted like a process where one particle exits the process unscattered. This process factorizes out the overlap

$$\langle \theta \,|\, \theta \rangle \tag{4.99}$$

which, as common in quantum mechanics, is a $\delta(0)$ in infinite volume. On the other hand, if no such overlap takes place, it is possible to use the crossing symmetry of the theory to consider the *out* state as the analytic continuation of an *in* state, where one particle has been brought to imaginary part equal to $i\pi$. We have then

$$\begin{aligned} {}_{\bar{a}_1}\langle \theta_1 | \, \mathcal{O} \, | \theta_2 \dots \theta_n \rangle_{a_2 \dots a_n} &= \langle 0 | \, \mathcal{O} \, | \theta_1 + i\pi, \theta_2 \dots \theta_n \rangle_{a_1, a_2 \dots a_n} \\ &\quad + \sum_{k \neq 1} 2\pi \delta(\theta_1 - \theta_k) \langle 0 | \, \mathcal{O} \, | \theta_1 \dots \hat{\theta}_k \dots \theta_n \rangle \end{aligned} \tag{4.100}$$

with the $\hat{}$ denoting exclusion.

- Kinematic pole relation
  As a consequence of the crossing and of (4.94,4.97), one has

$$-i \, \mathrm{Res}_{\theta' \to \theta} \, F^{\mathcal{O}}_{n+2}(\theta' + i\pi, \theta, \theta_1 \dots \theta_n) = \left(1 - e^{2\pi i \gamma_{\phi_a} \mathcal{O}} \prod_{k=1}^{n} S(\theta - \theta_k) \right) F^{\mathcal{O}}_n(\theta_1 \dots \theta_n) \tag{4.101}$$

  where for notational simplicity, only one particle kind has been considered and the number of particles has been explicitly indicated. This is a recursion relation for form factors, allowing to express $\mathcal{F}_n$ in terms of $\mathcal{F}_n$.

- Bound state pole relation
  Suppose the two-particle scattering matrix has a pole like (4.65)

$$S_{a_n, a_{n+1}}(\theta) \sim \frac{i \left(\Gamma^b_{a_n, a_{n+1}} \right)^2}{\theta_n - \theta_{n+1} - i u^b_{a_n, a_{n+1}}} \tag{4.102}$$

  identifying a bound state. Then, as a consequence of (4.94), the form factors inherit bound state poles from the S-matrix

$$\mathcal{F}^{\mathcal{O}}(\theta_1, \dots, \theta_{n+1})_{a_1, \dots, a_{n+1}} \sim \frac{i \Gamma^b_{a_n, a_{n+1}}}{\theta_n - \theta_{n+1} - i u^b_{a_n, a_{n+1}}} \mathcal{F}^{\mathcal{O}}(\theta_1, \dots, \theta_{n-1}, \theta)_{a_1, \dots, a_{n-1}, b} \tag{4.103}$$

  which relates different families of form factors.

Altogether, the conditions (4.93-4.103) define the boundary conditions of a Riemann-Hilbert problem for the unknown function $\mathcal{F}$ of an assigned number of variables. It is then on physical grounds that different families of solutions are put in correspondence with given operator and interpolating field.

It has been shown in a series of works (see, for instance [105]) that the spectral representation for massive theories has a fast convergent behaviour, therefore accurate estimates of correlators can be obtained by just using the first few exact terms in the form factor

expansion of the correlation function. This feature can be ultimately tracked down to the peculiar behaviour of the n–particle phase space in two dimension [98], when such particles are *massive*.

However, not much of this analytic structure is left in finite size L. In facts, not all the combinations of rapidities are allowed in the form factors (4.92), which makes problematic their determination. Moreover, being proportional to the volume, kinematical poles are expected to be smoothed out. In addition to this, whenever we cannot neglect exponential corrections in the size $e^{-\mu L}$, with μ of the order of the mass of the lightest excitation, the masses of asymptotic particles themselves are corrected, which forces to reconsider (4.103,4.94). This will be explored in the next section.

4.3 Field theories in finite volume

We are interested in studying the effects of having an integrable quantum field theory defined on a finite geometry. The simplest situation is the one in which one dimension is compact and runs on a circle of size L, in other words, when we study the theory on a cylinder.

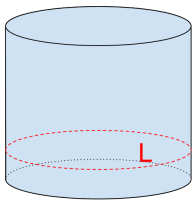


Figure 4.6:

Finite size effects are known to play a fundamental role in statistical physics and are widely studied [106]. In particular, in a certain interval around the critical temperature of a second order phase transition, the divergence of the correlation length is necessarily smoothed by the limited geometry. Analogously, the asymptotic behaviour of correlation functions at the critical point is changed from a power law to an exponential decay by the introduction of a length scale [107]. In facts, it is known that no phase transition takes place in a finite volume; nevertheless, information about the critical theory, such as the conformal charge, of lattice models can be obtained from the finite–size scaling of thermodynamic quantities.

In the approach of section 4.1 to the low-energy description of one-dimensional quantum models, when these are defined on a finite ring, finite size effect play a major role [108].

The interest also extends to the study of these systems in the scaling limit at finite *temperature*. The definition of a quantum field theory at finite temperature is based on the correspondence with statistical mechanics: a generic partition function for a quantum system with Hamiltonian H with one or many degrees of freedom is

$$Z(L) = Tr\left[e^{-LH}\right] \tag{4.104}$$

and can be written as

$$\int dq \langle q|e^{-LH}|q\rangle \tag{4.105}$$

in which the integration runs over all the possible configurations $|q\rangle$ of the system, hence it represents a multiple integral if there are many degrees of freedom. The expression above represents the evolution along an imaginary time direction of the state $|q\rangle$ according to the Hamiltonian H, projected on the initial state $\langle q|$ and then summed over all the possible initial configurations $|q\rangle$. This can be written in a path-integral formalism as:

$$\int_{q(L)=q(0)} \mathcal{D}q(\tau) e^{-\int_0^L d\tau \int dx \mathcal{L}_E} \tag{4.106}$$

When the number of degrees of freedom goes to infinity, the latter can be used to define a quantum field theory at finite temperature $1/L$.

In facts, that there are two different ways of defining a quantum theory on a cylinder. On the one hand, the infinite direction can be chosen as quantization line and the evolution of the corresponding space of states takes place along the $-x$ direction, where the sign is chosen for preserving frame orientation. In this case the space would be infinite, but the (imaginary) time is periodic. On the other hand, one can consider the field theory states on the circle and the vertical axis plays the role of time direction. From the field theory point of view, the two ways of quantizing the theory generically amount to different labels for space and time directions in the Lagrangian and are therefore equivalent.

In the finite–size case, the allowed momenta of one-particle states are quantized according to

$$e^{i\,L\,m\sinh\theta} = (-1)^F$$

with F the fermion number. For multiparticle states, a similar quantization condition should be written, keeping into account the scattering of a particle with all the others. This is straightforwardly done for diagonal S-matrices

$$e^{i\,L\,m_{a_1}\sinh\theta_1} S_{a_1,a_2}(\theta_{1,2}) S_{a_1,a_3}(\theta_{1,3}) \dots S_{a_1,a_M}(\theta_{1,M}) = (-1)^F \tag{4.107}$$

and similarly for all the other particles. These take the name of *Bethe-Yang equations*. In the nondiagonal case, the complete scattering process should be diagonalized, which results in a more sophisticated treatment (see e.g. [109]).

Integrability of certain $1+1$–dimensional quantum field theories allows to computed exactly the finite–volume ground state energy (Casimir energy) from the thermodynamic approach [110] for theories with diagonal S–matrices or kink–like particle content, exploiting the Bethe-Yang quantization. For the sine-Gordon model, instead, we need to first introduce the nonlinear integral equation formalism, which will be done in section 5.1.3.

In an interacting quantum field theory, each particle can be seen as surrounded by a vacuum polarization cloud of virtual particles. If we put the particle in a box of linear size L, the cloud is somehow compressed into the box, and finite-size mass shifts become appreciable as soon as the cloud becomes of the same size as the box.

The leading finite mass shift can be computed exactly [111] and is constituted by two contributions. The first one arises from all the diagrams in which the worldlines of the virtual particles of the cloud wind around the cylinder once before annihilating. This is called F–term in the formula of mass correction. The second contribution comes from virtual processes in which the original particle decays into two constituents, which travel around the compact direction before recombining to give back the original particle. This is called μ–term.

Then both the particle content of the theory and the amplitude and its dynamical properties enter the determination of the mass shifts [112]. The mass correction formula take a simpler form in $1+1$ dimensions [113], where it can be written in terms of the exact two-particle S-matrix:

$$\begin{aligned}\Delta m_a(L) &= -{\sum_{b,c}}' \Theta(m_a^2 - |m_b^2 - m_c^2|)\mu_{abc}(\Gamma_{ab}^c)^2 \\ &\quad -{\sum_{b}}' \mathcal{P}\int \frac{d\theta}{2\pi} m_b \cosh\theta \left(S_{ab}^{ab}(\theta + i\pi/2) - 1\right) e^{-m_b L\cosh\theta} + O\left(e^{-\sigma_a L}\right)\end{aligned} \tag{4.108}$$

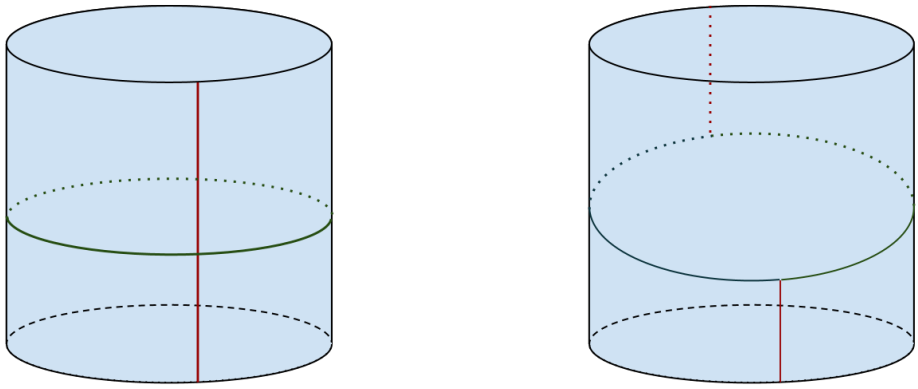


Figure 4.7: Left: one virtual particle is emitted and travels across the cylinder before annihilating (F–terms). Right: an asymptotic particle decays into two virtual particles, which recombine on the other side of the cylinder (μ–terms).

where Θ is the step function, $\mu_{abc} = m_b \sin u_{ab}^c = m_c \sin u_{ac}^b$ and we assume that $\Gamma_{ab}^c = 0$ if the particle c is not a bound state of a and b, i.e., if there is no bound state pole in the c channel in the $a + b$ scattering amplitude. The prime sign reminds not to include in the sum any contribution which is smaller than the error term. This error term is estimated in [113] as $\sigma_a \precsim \mu_{a11}$. We recognise in the first line the μ–terms and in the second the F–terms contributions to the mass corrections, as depicted in figure 4.7.

As we saw in section (4.2.3), the problem of computing correlation functions on the Euclidean plane can be solved in a very elegant way by the standard form factor approach. Conversely, for finite size, while studies of form factors have been carried on in semiclassical approximation in [114, 115], the lack of a simple expression for the vacuum of the theory and its excitations still constitutes an obstacle for the method.

A possible way to tackle the problem may be that of considering the system at finite temperature. With this setting, it is possible to use the "infrared" form factors, but a suitable regularization is nonetheless necessary. One-point functions can be efficiently computed by series [116, 117], while the formalism for two-point functions, perturbative in the exponential corrections in the volume, relies on a double series expansion containing suitably regularized infinite-volume form factors [118].

Going back to the original perspective, it is known [111] that corrections to the particle masses which constitute the spectrum in the infrared limit are exponentially suppressed in the system size. This fact allowed the proposal [119] of a finite-volume formalism, based on the infinite-volume form factors, that is correct up to terms which are exponentially suppressed in the volume. The method has been applied to the sine-Gordon theory in [120, 121, 122] to soliton and breather form factors. An alternative regularization has been proposed in [123] and applied to the computation of the dynamical structure factor in prototypical integrable field theories, although it is presently unclear how to extend the scheme to more general form factors.

Chapter 5

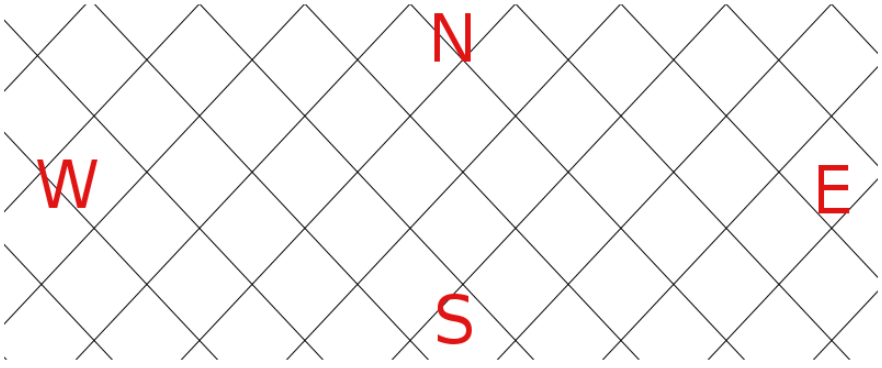
Correlation functions in finite volume from the light-cone lattice regularization

This chapter presents the problem of computing correlation functions of the integrable sine-Gordon field theory. We introduce a novel approach, by exploiting the algebraic construction of its (known) lattice regularization. Necessary explanations, which range from the definition of the lattice to the identification of the necessary operators, are provided in section 5.1, while the original contribution is presented in section 5.2.

5.1 The light-cone lattice approach

In this section, we will present the construction of [124], which shows how the massive Thirring field theory can be discretized on a properly defined lattice. We shall pay some attention to introduce this construction, since it is necessary for our purposes to identify the lattice representation of the sine-Gordon field operator, as well as for fixing the notation. We will then use the correspondence at the operator level of the $U(1)$ charge in sine-Gordon and Thirring models.

Let's begin by considering a square lattice, rotated by 45 degrees, as in the figure below

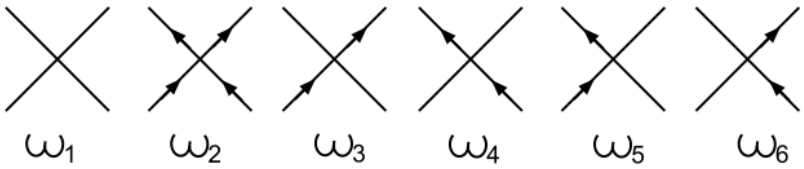


This will be referred to as a light-cone lattice. One can think to this lattice as a discretization of a two-dimensional Euclidean spacetime, by interpreting each site as an event and by as-

signing time to the $S \to N$ direction and space to the $W \to E$ direction. A fermionic variable will be defined on the links (which go either in the NW or in the NE direction).

The dynamics along the lattice links naturally defines a light-cone, or a set of events which can be causally connected, both in the past and in the future. At each time step, a particle can move on the edge of its light-cone either to the left with fixed "bare" rapidity $-\Lambda$ or to the right with rapidity Λ, so that we can divide them into left (L) or right (R) movers. A local R-matrix is assigned to each lattice site and represents the scattering of left-movers on right-movers. If unscattered, a particle will continue to move to the left or to the right with lattice rapidity $\pm\Lambda$, which therefore represents the maximum allowed, i.e., an UV cutoff for the theory.

By imposing that the sum of ingoing particles, i.e. in the SW and SE directions from each vertex, is equal to the sum of outgoing particles in the NE and NW directions one exactly has an "ice rule" [2], which is equivalent to a requirement of conservation of the number of particles. To be clear, by denoting $|\uparrow\rangle = \textit{occupied}$ and $|\,|\,\rangle = \textit{empty}$, the allowed configurations are



These configuration are invariant under the $U(1)$ transformation

$$|\,|\,\rangle \to e^{i\lambda}\,|\,|\,\rangle \qquad \text{and} \qquad |\uparrow\rangle \to e^{-i\lambda}\,|\uparrow\rangle \tag{5.1}$$

for some real λ, whose associated charge is indeed the particle number (see 5.1.2 below).

Without loss of generality, one can naturally assume the normalization of the vacuum-to-vacuum amplitude to be $\omega_1 = 1$. Moreover, symmetry under parity ($L \leftrightarrow R$) implies $\omega_5 = \omega_6 = c$ and $\omega_3 = \omega_4 = b$, while unitarity imposes

$$|b|^2 + |c|^2 = 1 \qquad b\,c^* + b^*\,c = 0 \tag{5.2}$$

so that the local $\check{R}$ matrix assumes the form (1.4):

$$\check{R}(\lambda) = PR(\lambda) = \begin{array}{ccc} \alpha & & \beta \\ & \times & \\ \alpha' & & \beta' \end{array} = \begin{pmatrix} 1 & & & \\ & c(\lambda) & b(\lambda) & \\ & b(\lambda) & c(\lambda) & \\ & & & 1 \end{pmatrix} \tag{5.3}$$

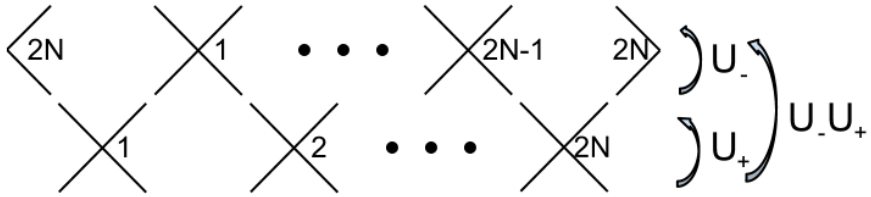
in which a generic dependence from a complex parameter λ has been inserted and we have also assumed that the vacuum-to-vacuum amplitude is equal to the particle-to-particle amplitude.

According to the discussion of Section 1.1, we can make sure that this system is *integrable* at the lattice level by requiring that the local weights satisfy the Yang-Baxter relation (1.2). The requirement of relativistic invariance of the scaling limit implies that this scattering is a function only of the lattice rapidity difference $\Lambda - (-\Lambda)$, which is constant through the lattice, so that the local "weights" of the vertexes depend on a single parameter.

5.1.1 Dynamics on the light-cone

Each link represents a possible portion of the particle wordline. A state of the system at a given time will be represented as a vector $|\alpha_1, \ldots, \alpha_{2N}\rangle$ containing the occupation numbers $\alpha_j = 0, 1$ of each link intersecting an horizontal line at a given vertical position.

The elementary dynamics on the light-cone lattice is generated by the operators U_+ and U_-, which act as in figure:



The operator U_+ connects the, say, odd lines of the vertex model to the even ones, which are displaced in the time direction by half a step in lattice units. On the other hand, the operator U_- connects the even lines to the odd ones, by producing a shift of $a/2$ in the vertical direction.

In terms of the microscopic amplitudes, they read

$$\langle \alpha_1, \ldots, \alpha_{2N} | \, U_+ \, | \alpha'_1, \ldots, \alpha'_{2N} \rangle \;=\; \prod_{n=1}^{N} R^{\alpha_{2n-1}, \alpha_{2n}}_{\alpha'_{2n-1}, \alpha'_{2n}} \tag{5.4}$$

$$\langle \alpha'_1, \ldots, \alpha'_{2N} | \, U_- \, | \alpha_1, \ldots, \alpha_{2N} \rangle \;=\; \prod_{n=1}^{N} R^{\alpha_{2n}, \alpha_{2n+1}}_{\alpha'_{2n}, \alpha'_{2n+1}} \tag{5.5}$$

The unit time translation operator can be chosen to be:

$$U = U_+ U_- \qquad\qquad U' = U_- U_+ \tag{5.6}$$

The one to be used depends on whether the initial state is on an odd or on an even line. In general, it acts like:

$$U \, | \alpha_1, \ldots, \alpha_{2N}, t \rangle = | \alpha'_1, \ldots, \alpha'_{2N}, t + a \rangle \tag{5.7}$$

and defines a light-cone transfer matrix.

One can similarly define a translation operator V, that operates a shift by $a/2$ in the horizontal direction, as:

$$V \, | \alpha_1, \ldots, \alpha_{2N}, t \rangle = | \alpha_{2N}, \alpha_1, \ldots, \alpha_{2N-1}, t \rangle \tag{5.8}$$

or, in terms of the permutation operators defined in (1.15), as:

$$V = P_{1,2} P_{2,3} \ldots P_{2N-1,2N} \tag{5.9}$$

so that V^2 is the space translation operator by one lattice unit step to the right and is a symmetry of the problem once we require periodic boundary conditions. Both $U_\pm$ and V are unitary operators and the commutation relations

$$\left[V^2, U_\pm \right] = 0 \qquad \Rightarrow \qquad \left[V^2, U^{(\prime)} \right] = 0 \tag{5.10}$$

hold. At the same time:

$$U_{\pm} = VU_{\mp}V^{\dagger}\,, \qquad U' = VUV^{\dagger} \tag{5.11}$$

It is natural to identify these operators as the exponentials of the lattice Hamiltonian and of the lattice momentum operators, generating a lattice unit time and space shift

$$U = e^{-iaH}\,, \qquad V = e^{-iaP} \tag{5.12}$$

It is moreover useful to introduce operators that implement transport in the light-cone directions. These are defined to be U_R and U'_R, that produce translations by $a/2$ in the vertical direction and by $a/2$ in the horizontal direction, and U_L (or U'_L), that implements translations by $a/2$ in the vertical direction and by $-a/2$ in the horizontal direction. In formulas:

$$U_R = VU_- = U_+V \qquad U_L = U_+V^{\dagger} = V^{\dagger}U_- \tag{5.13}$$

$$U'_R = VU_+ = U_-V \qquad U'_L = U_-V^{\dagger} = V^{\dagger}U_+ \tag{5.14}$$

It should be clear that

$$U_L = e^{-ia(H-P)/2} \qquad U_R = e^{-ia(H+P)/2} \tag{5.15}$$

5.1.2 Lattice fermion fields

We can represent in our model of fermions the group relations of the previous section, by defining a fermion field ψ_n defined on the links of the model. It turns out to be convenient to use both the equivalent notations:

$$\psi_{L,n} = \psi_{2n-1} \qquad \psi_{R,n} = \psi_{2n} \tag{5.16}$$

where we defined *left* movers on the odd sites and *right* movers on the even ones. A reference state at a given time is given by $|0,\dots,0\rangle$. Lattice fields will act on this as

$$\psi^{\dagger}_{n_1}\dots\psi^{\dagger}_{n_M}|0,\dots,0\rangle = |0\dots 1_{n_1},\dots 0\dots 1_{n_M}\dots 0\rangle \tag{5.17}$$

The local R-matrix in (5.3) is a representation on the basis $\{|0\rangle_n\,,|1\rangle_n\}$ of the operator

$$R_{n,m} = \mathbf{1} + bK_{n,m} + (c-1)K^2_{n,m} \tag{5.18}$$

with

$$K_{n,m} = \psi^{\dagger}_n\psi_m + \psi^{\dagger}_m\psi_n \tag{5.19}$$

and reduces to the exchange operator (1.15) for special values of the parameters, such that $b=1,\ c=0$:

$$P_{n,m} = \mathbf{1} + K_{n,m} + K^2_{n,m} \tag{5.20}$$

An essential point for the subsequent discussion, is that the fermion operators on a given time slice can be constructed by means of spin operators through a standard Jordan-Wigner transformation:

$$\psi_n = \sigma^-_n \prod_{l=1}^{n-1}\sigma^z_l \qquad \psi^{\dagger}_n = \sigma^+_n \prod_{l=1}^{n-1}\sigma^z_l \tag{5.21}$$

This implies that the $\check{\mathrm{R}}$-matrix can be written in spin language as

$$\check{\mathrm{R}}_{m,n}(\lambda) = \frac{1+c}{2}\,\mathbf{1}_{m,n} + \frac{1-c}{2}\sigma_m^z\sigma_n^z + \frac{b}{2}\,(\sigma_m^x\sigma_n^x + \sigma_m^y\sigma_n^y) \tag{5.22}$$

which allows us to exploit the discussion of Chapter 1 about integrable spin models.

An important consequence of the definition of the light-cone evolution operators $U_{L/R}$ (5.13) and of the representations (5.4) and (5.9) is that

$$\begin{aligned} U_R\psi_{2n-2}U_R^\dagger &= U_L\psi_{2n}U_L^\dagger = \check{\mathrm{R}}_{2n-1,2n}\,\psi_{2n-1}\,\check{\mathrm{R}}^\dagger_{2n-1,2n} \\ U_R\psi_{2n-1}U_R^\dagger &= U_L\psi_{2n+1}U_L^\dagger = \check{\mathrm{R}}_{2n-1,2n}\,\psi_{2n}\,\check{\mathrm{R}}^\dagger_{2n-1,2n} \end{aligned} \tag{5.23}$$

which allow, together with the definition (5.18), the computation of the lattice field equations

$$U_L\psi_{2n}U_L^\dagger = c^*\psi_{2n-1} + b^*\psi_{2n} - (b^*+b)N_{2n-1}\psi_{2n} + (c-c^*)\psi_{2n-1}N_{2n} \tag{5.24}$$

$$U_R\psi_{2n-1}U_R^\dagger = b^*\psi_{2n-1} + c^*\psi_{2n} - (b^*+b)\psi_{2n-1}N_{2n} + (c-c^*)N_{2n-1}\psi_{2n} \tag{5.25}$$

where the local number operator

$$N_n = \psi_n^\dagger\psi_n \tag{5.26}$$

has been introduced. These lattice field equations are *local*, involving only products of operators on neighbouring links. Moreover, they contain products of up to three operators, which suggests that they may be seen as generated from an Hamiltonian with a quartic interaction. In facts, one can regard the linear part of (5.24) as a "free" part and the cubic part as deriving from interactions. By requiring c to be real and b to be purely imaginary one recovers the noninteracting, massless case.

A reduction analogous to (5.23) holds also in the case of the bilinear field (5.26). Explicit computation allows to check that

$$\begin{aligned} U_LN_{2n}U_L^\dagger = U_RN_{2n-2}U_R^\dagger &= N_{2n} + |c|^2(N_{2n-1}-N_{2n}) + b^*c(\psi_{2n-1}^\dagger\psi_{2n} + \psi_{2n-1}\psi_{2n}^\dagger) \\ U_RN_{2n-1}U_R^\dagger = U_LN_{2n+1}U_L^\dagger &= N_{2n-1} - |c|^2(N_{2n-1}-N_{2n}) - b^*c(\psi_{2n-1}^\dagger\psi_{2n} + \psi_{2n-1}\psi_{2n}^\dagger) \end{aligned} \tag{5.27}$$

A straightforward consequence is that the *bare fermion number* operator

$$Q = \sum_{n=1}^{2N} N_n \tag{5.28}$$

is conserved. To show it, it is sufficient to evolve $J_n = N_{2n-1} + N_{2n}$ either by U_L or by U_R and to sum over site indexes in (5.27). This charge, as we have seen, is associated to the invariance under the $U(1)$ transformation (5.1) of the lattice dynamics. In the sine-Gordon language, it is the *solitonic charge* (4.71), the number of solitons minus the number of antisolitons present in the state. In terms of the spin formulation associated with (5.22), it becomes

$$Q = \sum_{n=1}^{2N}\left(\frac{1}{2} + S_n^z\right) \tag{5.29}$$

which just counts the total number of up spins. We anticipate here that the normal ordering prescription for these operators amounts to the subtraction of their expectation value on the vacuum. It will be seen in section 5.1.3 that the vacuum can be specified in terms of the roots of the Bethe equations of this system. For the time being, we only need to state that it is and half-filled state in the fermionic formulation or, equivalently, has zero total spin in the spin chain formulation. In other words, the normal ordered (5.29) is the total spin operator.

Continuum limit

We now wish to identify the continuum limit of the lattice field equations (5.24) and define the interacting field theory according to such equations. The problem that is met when performing this operation is that any Hamiltonian identified in this way is to be considered heuristic, since in general it must be renormalized in the scaling process.

The strategy followed in [124] is then to consider a "bare" continuum limit, which is defined by letting the lattice spacing go to zero while keeping the size constant:

$$a \to 0 \,, \qquad N \to \infty \,, \qquad L = 2\,a\,N \tag{5.30}$$

In a second step, the spectrum of the theory are computed *exactly* at any given value of a, N, Λ and is therefore free of renormalization effects: from that, it is possible to fix the correct scaling of the parameters. With this at hand, it is possible to evaluate nonperturbatively the scattering amplitudes of the elementary excitations of the theory and show that it agrees with the scattering matrix of the sine-Gordon model [125].

With the scaling (5.30), the field has naïvely dimension $\sqrt{a}$ on the plane, i.e., $\psi_{L/R,n} \to \sqrt{a}\psi_{L/R}(x)$. This means that when $a = 0$ exactly, the (5.24) reduce to

$$\begin{aligned} U_{L/R}\psi(x)_L U^\dagger_{L/R} &= b^*\psi_L(x) + c^*\psi_R(x) \\ U_{L/R}\psi(x)_R U^\dagger_{L/R} &= b^*\psi_R(x) + c^*\psi_L(x) \end{aligned} \tag{5.31}$$

To conveniently identify the right and left movers in this limit, we can assume that $|c| = 1$ and $b = 0$ when $a = 0$, by which the local scattering amplitude is reduced to (5.20). We can then define the quantity

$$\lim_{a\to 0} \frac{b}{|b|} = e^{i\gamma} \tag{5.32}$$

and we refer to γ as *anisotropy parameter*, since it will be clear soon that it is exactly the one appearing in (1.5). Then the spaces at two neighbouring links are at a distance a, which implies that they will not be distinguishable in the thermodynamic limit and that

$$U_L(a=0) = U_R(a=0) = e^{-i\gamma Q} \,, \qquad Q = \int_0^L \left(\psi^\dagger_L\psi_L + \psi^\dagger_R\psi_R\right) \tag{5.33}$$

in which the operator Q is not normal ordered, yet. Analogously, we must have at least that $c \propto a$ to meet the above requirements. Unitarity (5.2) implies that the proportionality constant has a phase which is shifted by $\pi/2$ with respect to the one of b and one can set

$$c \sim -ie^{i\gamma}\frac{m}{2}a\left(1 + O(a^2)\right) \,, \qquad b = e^{i\gamma}\left(1 + O(a^2)\right) \tag{5.34}$$

where m is a free parameter of the theory and is the fermion mass parameter in the Lagrangian of the Thirring model and the mass of the elementary excitations in the sine-Gordon model.

One can define two generators of translation along the light-cone edges as

$$H_{L/R} = \lim_{a\to 0} \frac{1}{a}\left(e^{-i\gamma Q}U_{L/R} - 1\right) \tag{5.35}$$

in such a way that

$$H = H_R + H_L \qquad\qquad P = H_R - H_L \tag{5.36}$$

Using this definition, which amounts to the substitution $U_{R/L} \simeq e^{i\gamma Q}\left(\mathbf{1} - iaH_{R/L} + \ldots\right)$ in the lattice field equations, one obtains

$$\begin{aligned}
[H_R, \psi_{R,n-1}] &= \lim_{a\to 0} \frac{i}{a}|b|\Big[\psi_{R,n} - \frac{c}{b}\psi_{L,n} - \frac{1}{|b|}\psi_{R,n-1} - \left(1+e^{2i\gamma}\right)\Big(N_{L,n}\psi_{R,n} - \frac{c}{b}\psi_{L,n}N_{R,n}\Big)\Big] \\
[H_R, \psi_{L,n}] &= \lim_{a\to 0} \frac{i}{a}|b|\Big[\psi_{L,n} - \frac{c}{b}\psi_{R,n} - \frac{1}{|b|}\psi_{L,n} - \left(1+e^{2i\gamma}\right)\Big(\psi_{L,n}N_{R,n} - \frac{c}{b}N_{L,n}\psi_{R,n}\Big)\Big]
\end{aligned} \tag{5.37}$$

and analogous relations for the light-cone evolution of the other operators. Special care must be kept when performing the continuum limit of composite operators, such as bilinear fields containing operators at adjacent sites. In particular, one must remember that (5.16) implies

$$\psi_{R,n} \sim \sqrt{a}\psi_R(x+0)\,, \qquad \psi_{L,n} \sim \sqrt{a}\psi_L(x-0) \tag{5.38}$$

We refer the reader to the original paper [124] about this aspect stating only that the prescription is

$$\frac{1}{a}\psi_{R,n}\psi_{L,n} \to \frac{2}{1-e^{2i\gamma}}\left(\psi_R\psi_L\right)(x) \tag{5.39}$$

where the definition

$$\left(\psi_A\psi_B\right)(x) = \frac{1}{2}\left(\psi_A(x+0)\psi_B(x-0) - \psi_A(x-0)\psi_B(x+0)\right) \tag{5.40}$$

has been introduced. By substituting the latter into the two (5.37) and the two analogues with exchanged chirality, we obtain the continuum field equations

$$\begin{aligned}
[H_R, \psi_R(x)] &= i\frac{\partial}{\partial x}\psi_R(x) - \frac{m}{2}\psi_L(x) - g\left(\psi_L^\dagger\psi_L\psi_R\right)(x) \\
[H_R, \psi_L(x)] &= -\frac{m}{2}\psi_R(x) - g\left(\psi_R^\dagger\psi_R\psi_L\right)(x) \\
[H_L, \psi_R(x)] &= -\frac{m}{2}\psi_L(x) - g\left(\psi_L^\dagger\psi_L\psi_R\right)(x) \\
[H_L, \psi_L(x)] &= -i\frac{\partial}{\partial x}\psi_L(x) - \frac{m}{2}\psi_R(x) - g\left(\psi_R^\dagger\psi_R\psi_L\right)(x)
\end{aligned} \tag{5.41}$$

with the definition

$$g = -2\cot\gamma \tag{5.42}$$

being the Thirring coupling constant.

Finally, from the equation of motion (5.41), one can recognize the Hamiltonian of the massive Thirring model

$$H = H_L + H_R = \int_0^L \left[-i\Psi^\dagger\left(\gamma^5\partial_x + im\gamma^0\right)\Psi + \frac{g}{2}J_\mu J^\mu\right] \tag{5.43}$$

with mass m, coupling g and the matrices (4.87)

When changing the reference state from the state annihilated by all the ψ_n to a state where the sea of negative energy levels is filled up, it is necessary to normal order products of operators, removing the leading singularities of the bilinears $\psi_{L/R}^\dagger\psi_{L/R}$. The issue is discussed again in [124] and leads to the normal ordered version of (5.43).

The requirement of integrability (1.2) at the lattice level, together with the unitarity constraint, leaves the local R-matrices determined to be of the form (5.3,1.5). In particular, one has a local Hilbert space associated to every link which is intersected by an horizontal line, at any given time $t = t_0$. These links are alternatively associated with a left- and a right-mover, i.e., with a local lattice rapidity $\mp\Lambda$. The particular choice of the inhomogeneity parameters which defines the sine-Gordon light-cone lattice regularization is then

$$h_n = (-1)^{n+1}\Lambda \tag{5.44}$$

In the absence of interactions, these are the only allowed lattice rapidities. Conversely, in the presence of a nontrivial local R-matrix, states with different numbers of excitations, each characterized by a set of bare rapidities, will be allowed. The purpose is then to determine the eigenstates of the evolution generated by the Hamiltonian (5.43), which at the discrete level amounts to diagonalizing the lightcone transfer matrix 5.13.

Algebraic formulation

We will use here the spin formulation of the lattice model to cast the problem in the algebraic formalism. We follow the general procedure of Chapter 1 and write the monodromy matrix (1.17) for these inhomogeneities:

$$\begin{aligned} T_0(\lambda) &= e^{i\omega\sigma_0^z} R_{0,1}(\lambda-\Lambda+i\gamma/2)R_{0,2}(\lambda+\Lambda+i\gamma/2)\dots R_{0,2N}(\lambda+\Lambda+i\gamma/2) \\ &= e^{i\omega\sigma_0^z} L_{0,1}(\lambda-\Lambda)L_{0,2}(\lambda+\Lambda)\dots R_{0,2N}(\lambda+\Lambda) \end{aligned} \tag{5.45}$$

we remind the reader the expression (1.11) and write the associated transfer matrix [126] in terms of

$$l(\lambda) = PL(\lambda) \tag{5.46}$$

as

$$\tau(\lambda) = \mathrm{Tr}_0\left[e^{i\omega\sigma_0^z} P_{0,1}l_{0,1}(\lambda-\Lambda)P_{0,2}l_{0,2}(\lambda+\Lambda)\dots P_{0,2N}l_{0,2N}(\lambda-\Lambda)\right] \tag{5.47}$$

in which $l(\lambda)$ has the property

$$l(-i\gamma/2) = PR(0) = \mathbf{1} \quad , \qquad l(-\lambda-i\gamma/2) = l(\lambda-i\gamma/2)^{-1} \tag{5.48}$$

Specializing the argument and exploiting this property, one sees that

$$\begin{aligned} \hat{\tau}(-\Lambda-i\gamma/2) &= \mathrm{Tr}_0\left[e^{i\omega\sigma_0^z} P_{0,1}l_{0,1}(-2\Lambda)P_{0,2}P_{0,3}l_{0,3}(-2\Lambda)\dots P_{0,2N}\right] \\ &= P_{2N-1,2N}\dots P_{1,2}\check{R}_{1,2}(-2\Lambda)\dots\check{R}_{2N-1,2N}(-2\Lambda) = V^\dagger U_+^\dagger = U_R^\dagger = e^{i\frac{a}{2}(H+P)} \end{aligned} \tag{5.49}$$

and

$$\begin{aligned} \hat{\tau}(\Lambda-i\gamma/2) &= \mathrm{Tr}_0\left[P_{0,1}P_{0,2}l_{0,2}(2\Lambda)P_{0,3}\dots P_{0,2N}l_{0,2N}(2\Lambda)\right] \\ &= P_{2N-1,2N}\dots P_{1,2}\check{R}_{2,3}(2\Lambda)\dots\check{R}_{2N,1}(2\Lambda) = V^\dagger U_- = U_L = e^{-i\frac{a}{2}(H-P)} \end{aligned} \tag{5.50}$$

apart from a proportionality constant. This shows that the eigenstates of the evolution operator U generated by the lattice version the sine-Gordon/massive Thirring model are those which diagonalize the light-cone transfer matrices derived from the spin monodromy matrix (5.45) [124, 126].

5.1.3 Bethe roots in the thermodynamic limit

As we have seen at the beginning of the section, the regularization features another parameter, namely the rapidity of a particle moving on the light-cone Λ, whose scaling has not been fixed yet. It will be done in this section, while studying the eigenvalues of the transfer matrix in the thermodynamic limit.

We rewrite here the system of equations (1.51)relative to the eigenstate identified by the roots $\{\mu\}$, for the above choice of the inhomogeneities, in the form:

$$B(x|\{\mu\},\omega) = B^{\omega}_{\mu}(x) = \frac{a(x)}{d(x)} \prod_a \frac{\sinh(x-\mu_a+i\gamma)}{\sinh(x-\mu_a-i\gamma)} e^{2i\omega} \tag{5.51}$$

with, as in 1.44:

$$\begin{aligned} a(x) &= \left[\frac{\sinh(x-\Lambda-i\frac{\gamma}{2})\sinh(x+\Lambda-i\frac{\gamma}{2})}{\sinh^2(x-\Lambda)}\right]^N \\ d(x) &= \left[\frac{\sinh(x-\Lambda+i\frac{\gamma}{2})\sinh(x+\Lambda+i\frac{\gamma}{2})}{\sinh^2(x-\Lambda)}\right]^N \end{aligned} \tag{5.52}$$

The Bethe equations are then written as:

$$B^{\omega}_{\mu}(\mu_j) = -1, \quad j = 1,\dots,M \tag{5.53}$$

In the following, it will turn out to be useful the use of the rescaled variables (the "rapidities" of the field theory) and inhomogeneities, and in particular of:

$$\theta_j = \frac{\pi}{\gamma}\mu_j \ , \qquad \Theta = \frac{\pi}{\gamma}\Lambda \tag{5.54}$$

The logarithmic form of (5.51), i.e., of (5.53) when considered as a function of one rapidity μ_j, defines in this region the *counting function*:

$$Z(x) = -i \log B(\frac{\gamma}{\pi}x) \tag{5.55}$$

which takes its name from the fact that solutions of the logarithmic form of the Bethe equations are characterized by:

$$Z(\theta_j) = 2\pi I_j \qquad , \qquad I_j \in \mathbf{Z} + \frac{1+\delta}{2} \ , \qquad \delta = (N-S)_{mod2} \tag{5.56}$$

for some integer or half-integer I, with $\delta = 0, 1$ specifying the sector of the field theory the state belongs to, as explained in [127, 128, 129, 130]. In the perspective of section 4.1, this procedure is in a way analogue to the one used to associate a function to a configuration. For the roots on the real axis, the situation is exactly as depicted in 4.1; in addition, there can appear complex Bethe roots as well, yet a counting function can still be defined as a function of the complex plane through the same quantization condition.

Setting

$$\phi(\theta,\nu) = -i \log \frac{\sinh\frac{\gamma}{\pi}(i\pi\nu-\theta)}{\sinh\frac{\gamma}{\pi}(i\pi\nu+\theta)} \tag{5.57}$$

the counting function can be written as

$$Z(\theta) = N\left[\phi(\theta+\Theta,1/2)+\phi(\theta-\Theta,1/2)\right] - \sum_{k=1}^{M}\phi(\theta-\theta_k,1/2) + 2\omega \tag{5.58}$$

We specify that the branch cut of the logarithm runs along the negative real semiaxis and that it is taken in the fundamental determination (FD) in (5.57), with $-\pi < \Im(\log arg) < \pi$. Moreover, using the representations (5.50,5.49) for the lightcone evolution operators in terms of the transfer matrix, one has:

$$\begin{aligned} \mathcal{E}(\{\lambda\}) &= \frac{1}{a}\sum_{j=1}^{M}\left(\phi(\Theta+\lambda_j,1/2)+\phi(\Theta-\lambda_j,1/2)-2\pi\right) \\ \mathcal{P}(\{\lambda\}) &= \frac{1}{a}\sum_{j=1}^{M}\left(\phi(\Theta+\lambda_j,1/2)-\phi(\Theta-\lambda_j,1/2)\right) \end{aligned} \tag{5.59}$$

for the energy and momentum of a state defined by the set $\{\lambda\}$ of roots.

The Bethe roots are obtained as zeros of the equation

$$1+(-1)^{\delta}e^{i\,Z(\lambda)}=0 \tag{5.60}$$

Solutions can be classified according to their position in the complex plane:

- *real roots* of the equation (5.60) constitute the Dirac sea in the field theory limit. They will be labelled by using a tilde on the variable and their set by $\{\tilde{\lambda}\}$.
- *holes* are real solutions of (5.60) that are not among the Bethe roots. Their number is N_H.
- *special* solutions are real solutions of (5.60) in which the counting function has a negative derivative ($Z' < 0$). Their number is denoted by M_S.
- *close* roots are present in pairs and have imaginary part in the strip between $-\min(p,1)\pi$ and $\min(p,1)\pi$. Their number is M_C.
- *wide* roots are present in pairs and have the absolute value of the imaginary part in the strip between $\min(p,1)\pi$ and $\pi(p+1)/2$. There will be M_W wide roots.
- *self-conjugated* roots sit on the boundary of the periodicity strip and have imaginary part equal to $\frac{\pi}{2}(p+1)$.

The analytic structure, and in particular the width of the fundamental strip, of the counting function may or may not coincide with the one of ϕ, depending on whether the roots in the state are all real or there appear complex pairs.

Elementary excitations and conserved charges

Quite remarkably, the conserved charges of the "infrared" sine-Gordon field theory are also correctly reproduced by the light-cone regularization. In order to see this, one has to evaluate the scaling limit of the perturbed model in such a way that the physical size of the system $L \to \infty$ together with the number of lattice sites. In this section, we show how the computation [125] works for the repulsive regime $\gamma < \pi/2$.

For the antiferromagnetic ground state, one can argue from the quantization of rapidities in the thermodynamic limit (see below) that the separation between two successive roots scales like $\sim 1/L$. This means that in infinite volume, roots are conveniently described by means of a density $\rho(\theta) = \frac{1}{2\pi N}Z'(\theta)$ of solutions. The latter can be obtained in the ground

state directly from the definition by using (5.58) and transforming the sum on the RHS in an integral.

$$\rho(\theta) = \frac{1}{2\pi}\left[\phi'(\theta+\Theta) + \phi'(\theta-\Theta)\right] - \int d\,\theta'\,\frac{1}{2\pi}\phi'(\theta-\theta')\rho(\theta') \tag{5.61}$$

Then, solving explicitly for ρ by Fourier transform, one can see that:

$$\rho(\theta) = \int \frac{dk}{2\pi} e^{ik\,\theta}\frac{\cos(k\Theta)}{\cosh\frac{\gamma k}{2}} \tag{5.62}$$

The simplest excitation is constituted by one hole in the Dirac sea. Due to the interacting nature of the system, the removal of one roots from the Dirac sea determines a variation of the whole root distribution. This can be taken into account by explicitly inserting the term containing the hole position

$$\rho(\theta) = \frac{1}{2\pi}\left[\phi'(\theta+\Theta) + \phi'(\theta-\Theta)\right] - \int d\,\theta'\,\frac{1}{2\pi}\phi'(\theta-\theta')\rho(\theta') + \frac{1}{2\pi N}\phi'(\theta-h) - \frac{1}{N}\delta(\theta-h) \tag{5.63}$$

in which the last term arises from the removal of the root at $\theta = h$ [131, 132]. One can solve then for the variation in the particle density

$$\delta\rho_h(\theta) = \int dq e^{iq\,\theta}\frac{2\pi}{2\pi+\tilde{\phi}'(q)}\left(\frac{1}{2\pi}\tilde{\phi}'(q) - 1\right) \tag{5.64}$$

This expression is interesting in connection with the excitation energy, which we can compute by using (5.59).

$$\mathcal{E}_h - \mathcal{E}_0 = \int d\,\theta\,[\phi(\Theta-\theta) + \phi(\Theta+\theta) - 2\pi]\,\delta\rho_h(\theta) = \frac{2}{a}\arctan\frac{\cosh\pi h/\gamma}{\sinh\pi\Lambda/\gamma} \tag{5.65}$$

while its momentum reads

$$\mathcal{P}_h = \int d\,\theta\,[\phi(\Theta-\theta) - \phi(\Theta+\theta)]\,\delta\rho_h(\theta) = -\frac{2}{a}\arctan\frac{\sinh\pi h/\gamma}{\sinh\pi\Lambda/\gamma} \tag{5.66}$$

From these expression we can fix the scaling of the inhomogeneity parameter Λ with respect to the lattice spacing. In facts, from the expansion

$$\arctan x = x - \frac{x^3}{3} + O(x^5) \tag{5.67}$$

we see that, making use of the rescaled rapidity $\theta_h = \pi h/\gamma$, a relativistic spectrum is associated to the hole

$$P_0(\theta_h) = m\cosh\theta_h \qquad\qquad P_1(\theta_h) = m\sinh\theta_h \tag{5.68}$$

This is true only in the double scaling limit

$$\Theta = \frac{\pi}{\gamma}\Lambda \quad , \quad \Theta\to\infty \quad , \quad a\to 0 \quad , \quad a\,e^{\Theta} = const = \frac{4}{m} \tag{5.69}$$

with m being the soliton mass (4.81). This fixes the scaling of the last parameter of the lattice regularization [125]. Analogous treatment can be carried on for multiparticle states, and allows to identify the kind of excitation (solitons, antisolitons, breathers...) from their scattering properties. We shall come back to this point later on.

It is also possible to compute [125] the eigenvalues of the transfer matrix $\hat{\tau}(\lambda)$, as defined from (5.45) in the infinite volume limit. The eigenvalue $\tau(\lambda)$ of $\hat{\tau}(\lambda)$ on a generic $k-$particle state can be read off as

$$\tau(\frac{\gamma}{\pi}\lambda) = \exp\left\{-2i\sum_{n=1}^{k}\arctan e^{\lambda-\theta_n}\right\} \tag{5.70}$$

where the $\theta_n = \frac{\pi}{\gamma}\lambda_n$ are the rescaled (physical) particle rapidities. As we have seen in Chapter 1, one can obtain a set of commuting charges from the expansion of the logarithm of the transfer matrix in the rapidity parameter. In this case, since the cutoff rapidity $\Theta \to \infty$, we obtain operators which commute with the evolution operators generated by the transfer matrix by expanding around $\lambda = \pm\infty$, which will be nonlocal with respect to the spin chain sites. Applying this procedure to the eigenvalues, with expansion parameter $z = e^{\mp\pi\lambda/\gamma}$, we have that:

$$\pm\, i\log\tau(z) = \sum_j \left(\frac{(-1)^j}{j+1/2}\sum e^{\pm(2j+1)\theta_n}\right) z^{2j+1} \tag{5.71}$$

From this expression, one can reconstruct the identity in operator form:

$$\pm\, i\log\hat{\tau}(\lambda) = \sum_{j=0}^{\infty}\left(\frac{4z}{m}\right)\hat{I}_j^{\pm} \tag{5.72}$$

For instance the combinations of the light-cone energy and momentum $I_0^+ \pm I_0^-$ yield the energy and momentum of the multisoliton solution. Analogously, eigenvalues of the higher conserved charges are reproduced:

$$I_j^{\pm} = \frac{(-1)^j}{j+1/2}\sum_{m=1}^{k}\left(\frac{m}{4}e^{\pm\theta_n}\right)^{2j+1} \tag{5.73}$$

and can be compared with the general eigenvalues (4.50) and with the results of [133] for sine-Gordon.

The nonlinear integral equation and a useful summation formula

Given that the twist parameter ω is a real number, the counting function Z is real analytic

$$Z(\theta^*) = Z(\theta)^* \tag{5.74}$$

In order to derive a single equation for Z in the thermodynamic limit, a strategy has been set in [125, 134, 135, 127]. Details of the derivation, that we briefly summarize here below, have been provided in [136]. To avoid dealing with logarithmic cuts, it is convenient to take the derivative of the counting function and to separate the contributions coming from the different kinds of roots:

$$\begin{aligned} Z'(\theta) &= N\left[\phi'(\theta+\Theta,1/2)+\phi'(\theta-\Theta,1/2)\right] - \left(\sum_r^{M_R}+\sum_r^{N_H}\right)\phi'(\theta-\lambda_r) \\ &\quad + \sum_h^{N_H}\phi'(\theta-\lambda_h) - \sum_c^{M_C}\phi'(\theta-\lambda_c) - \sum_w^{M_W}\phi'(\theta-\lambda_w) \end{aligned} \tag{5.75}$$

Assuming that a function f is analytic in a connected region of the complex plane, that contains all the roots $\{\lambda\}$, satisfying (5.53), we want to compute the sum of the function when evaluated in the position of the roots. To begin, we write it as:

$$\sum_j f(\lambda_j) = \oint \frac{du}{2\pi i} \frac{i\,(\text{-}1)^\delta\, Z'(u) e^{iZ(u)}}{1+(\text{-}1)^\delta\, e^{iZ(u)}} - \sum_h^{N_H} f(\lambda_h) + \sum_c^{M_C} f(\lambda_c) + \sum_w^{M_W} f(\lambda_w) - \sum_s^{N_S} f(\lambda_s) \quad (5.76)$$

in which the contour in the first term surrounds the real axis and no other root and accounts then for the sum over holes and real roots. Holes are subtracted by the second term, while the other families of roots are separately added by the successive summations. Then we make use of the simple relation:

$$\frac{i\,(\text{-}1)^\delta\, Z'(u^-)}{1+(\text{-}1)^\delta\, e^{-iZ(u^-)}} = iZ'(u^-)\left(1 - \frac{(\text{-}1)^\delta}{1+(\text{-}1)^\delta\, e^{iZ(u^-)}}\right) \quad (5.77)$$

to obtain

$$\begin{aligned}\sum_j f(\lambda_j) \;\; = \;\; & \int \frac{du}{2\pi} Z'(u^-) - \sum_{\sigma=\pm} \int \frac{du}{2\pi} \frac{(\text{-}1)^\delta\, Z'(u^\sigma)}{1+(\text{-}1)^\delta\, e^{-i\sigma Z(u^\sigma)}} \\ & - \sum_h^{N_H} f(\lambda_h) + \sum_c^{M_C} f(\lambda_c) + \sum_w^{M_W} f(\lambda_w) - \sum_s^{N_S} f(\lambda_s) \end{aligned} \quad (5.78)$$

By applying the above to the sums of ϕ' in (5.75), one can write

$$\begin{aligned}&\int \left[\delta(\theta-x) + \frac{1}{2\pi}\phi'(\theta-x)\right] Z'(x) = N\left[\phi'(\theta+\Theta,1/2) + \phi'(\theta-\Theta,1/2)\right] + \sum_h^{N_H} \phi'(\theta-\lambda_h) \\ &- \sum_c^{M_C} \phi'(\theta-\lambda_c) - \sum_w^{M_W} \phi'(\theta-\lambda_w) + \sum_{\sigma=\pm} \int \frac{dx}{2\pi} \phi'(\theta-x^\sigma) \frac{(\text{-}1)^\delta\, Z'(x^\sigma)}{1+(\text{-}1)^\delta\, e^{-i\sigma Z(x^\sigma)}} \end{aligned} \quad (5.79)$$

It is useful at this point to introduce the inverse of the integral operator $\mathbf{1}+\frac{\phi'}{2\pi}$ appearing in the left hand side of this equation. It is defined through its Fourier transform

$$\tilde{\Delta}(k) = \frac{1}{1+\frac{1}{2\pi}\tilde{\phi}'(k)} \quad (5.80)$$

Note that the convolution of Δ with ϕ' can be performed exactly by using the Fourier transform:

$$\tilde{\phi}'_\nu(k) = 2\pi \frac{\sinh\left(\frac{\pi}{2} - \nu\gamma\right) k}{\sinh\frac{\pi k}{2}} \quad (5.81)$$

assuming the argument is on the real axis. The result is:

$$\int dx \Delta(\theta-x)\phi'(x\pm\Theta,1/2) = \frac{1}{\cosh(\theta\pm\Theta)} \quad (5.82)$$

We hereby define the kernel by the integral

$$G(\theta) = \int \frac{dk}{2\pi} \frac{\tilde{\phi}'}{2\pi+\tilde{\phi}'} = (p+1)\int \frac{dk}{2\pi} \frac{\sinh\frac{p-1}{2}k}{\cosh\frac{\pi k}{2}\sinh\frac{\pi p k}{2}} \cos(\theta k) \quad (5.83)$$

The second expression holds as long as the imaginary part of the argument remains in the fundamental strip $|\Im\theta| < \min(1,p)\pi$, since the function (5.83) has a singularity in those points. Nevertheless, the definition is still valid also outside this domain, but special care must be taken when one performs the Fourier transform of (5.81) through the residue theorem and the imaginary part of the argument lies outside the above range. Then the function G_{II} is defined [125] for $|\Im\theta| > \min(1,p)\pi$:

$$2\pi G_{II}(\theta) = \begin{cases} \frac{i\,\mathrm{sgn}\,\Im\,\theta}{p}\left[\coth\frac{\theta}{p} - \coth\frac{\theta - i\pi\,\mathrm{sgn}\,\Im\,\theta}{p}\right] & p > 1 \\ i\,\mathrm{sgn}\,\Im\,\theta\left[\frac{1}{\sinh\theta} - \frac{1}{\sinh(\theta - i\pi p\,\mathrm{sgn}\,\Im\,\theta)}\right] & p < 1 \end{cases} \tag{5.84}$$

In terms of the function G, one can also write

$$\Delta(x) = \delta(x) - G(x) \tag{5.85}$$

Denoting $x^{\pm} = x \pm i\eta$, with η some small real quantity, the derivative (5.75) of the counting function can now be rewritten as

$$\begin{aligned} Z'(\theta) &= N\left[\frac{1}{\cosh(\theta+\Theta)} + \frac{1}{\cosh(\theta-\Theta)}\right] + \sum_{h=1}^{N_H} 2\pi G(\theta-\lambda_h) \\ &\quad - \sum_{c=1}^{M_C} 2\pi G(\theta-\lambda_c) - \sum_{w=1}^{M_W} 2\pi G(\theta-\lambda_w) - i\sum_{\sigma=\pm}\int dx G(\theta-x^{\sigma})\frac{i\,(\text{-}1)^{\delta}\,Z'(x^{\sigma})}{1+(\text{-}1)^{\delta}\,e^{-i\sigma Z(x^{\sigma})}} \end{aligned} \tag{5.86}$$

We now pause a bit on the last term of the above expression and connect it to the logarithmic derivative of the quantity $f(x+i\eta) = 1 + (\text{-}1)^{\delta}\,e^{iZ(x+i\eta)}$. It is possible that the latter may cross the branch cut of the logarithm, in which case the latter has a discontinuity $\pm 2\pi i$. This happens when

$$\Im Z(x+i\eta) < 0\,, \tag{5.87}$$

and

$$\Im e^{i\Re Z(x+i\eta)} = 0\,, \qquad \Re\,(\text{-}1)^{\delta}\,e^{i\Re Z(x+i\eta)} < 0 \tag{5.88}$$

By expanding in the shift from the real axis, one sees that the condition (5.87) is met for Z' when $\eta \to 0$, while (5.88) defines in the same limit a root (5.56). In other words, in the presence of *special* roots the logarithm of f in the fundamental determination has a jump discontinuity. However, the last ratio in (5.90) has no such a discontinuity, so that the correct manipulation (see [125, 136]) is

$$\pm\frac{(\text{-}1)^{\delta}\,iZ'(x^{\pm})}{1+(\text{-}1)^{\delta}\,e^{\mp iZ(x^{\pm})}} = \frac{d}{dx}\log\left(1+(\text{-}1)^{\delta}\,e^{\pm iZ(x^{\pm})}\right) \mp 2\pi i\delta(x-\hat{y}) \tag{5.89}$$

where $\hat{y}$ stands for a special root. We can then manipulate further the derivative of the counting function

$$\begin{aligned} Z'(\theta) &= N\left[\frac{1}{\cosh(\theta+\Theta)} + \frac{1}{\cosh(\theta-\Theta)}\right] + \sum_{h=1}^{N_H} 2\pi G(\theta-\lambda_h) - \sum_{c=1}^{M_C} 2\pi G(\theta-\lambda_c) - \sum_{w=1}^{M_W} 2\pi G(\theta-\lambda_w) \\ &\quad - \sum_{y}^{N_S} 2\pi\left(G(\theta-\hat{y}^{+}) + G(\theta-\hat{y}^{-})\right) - i\sum_{\sigma=\pm}\sigma\int dx G(\theta-x^{\sigma})\log'\left(1+(\text{-}1)^{\delta}\,e^{-i\sigma Z(x^{\sigma})}\right) \end{aligned} \tag{5.90}$$

With this expression and (5.89) at hand, the sum (5.78) may be written as

$$\begin{aligned}\sum_j f(\lambda_j) &= \int \frac{du}{2\pi}\left(\frac{N}{\cosh(u-\Theta)}+\frac{N}{\cosh(u+\Theta)}\right) f(u) - \sum_{k=1}^{N_h}(\Delta\star f)(h_k) \\ &+\sum_{k=1}^{M_c}(\Delta\star f)(c_k)+\sum_{k=1}^{M_w}(\Delta_{II}\star f)(w_k)+2\sum_{k=1}^{M_s}(\Delta\star f)(s_k) \\ &+i\sum_{\sigma=\pm}\int\frac{du}{2\pi}(\Delta\star f)(u^\sigma)\log'_{FD}\left(1+e^{i\sigma Z(u^\sigma)}\right) \end{aligned} \tag{5.91}$$

where the symbol $\star$ represents the convolution and the operator (5.80) appears, having used (5.85) with slight abuse of notation and the understanding that the δ, when integrated with a function, yields the function evaluated in the point that corresponds to the zero of its argument, even if such a zero is outside the real axis. By the aid of this expression, one can compute the scaling limit of the spin chain form factors.

By integrating (5.90) and performing the scaling limit (5.69), one obtains the nonlinear integral equation (NLIE) for the counting function:

$$Z(\theta) = m\,L\sinh\theta + g(\theta|\,\{I\}) - i\sum_{\sigma=\pm}\int dx G(\theta-x^\sigma)\log_{FD}\left(1+(-)^\delta e^{i\sigma Z(x^\sigma)}\right)+\alpha \tag{5.92}$$

with $\alpha=\frac{p+1}{p}\omega$ for neutral states and $\omega<\pi/2$.

$$\begin{aligned} g(\theta|\,\{I\}) &= \sum_h^{N_H}\chi(\theta-\lambda_h)-\sum_c^{M_C}\chi(\theta-\lambda_c) \qquad (5.93)\\ &\quad -\sum_s^{N_S}\left(\chi(\theta-\hat{y}^-)+\chi(\theta-\hat{y}^+)\right)-\sum_w^{M_W}\chi_{II}(\theta-\lambda_w) \\ \chi(\theta) &= 2\pi\int_0^\theta G(x)dx \qquad (5.94)\\ & \qquad (5.95)\end{aligned}$$

The subscript "II" reminds that the function χ must be considered in the second determination when the imaginary part of its argument exceeds $\min(1,p)\pi$, as in the case of wide roots. It is computed [136] from (5.84) as:

$$\chi_{II}(\theta)=\begin{cases} i\,\mathrm{sgn}\,\Im\theta\left[\log\sinh\frac{\theta}{p}-\log\sinh\frac{\theta-i\pi\,\mathrm{sgn}\,\Im\theta}{p}\right] & p>1 \\ i\,\mathrm{sgn}\,\Im\theta\left[\log\left(-\tanh\frac{\theta}{2}\right)+\log\left(\tanh\frac{\theta-i\pi p\,\mathrm{sgn}\,\Im\theta}{2}\right)\right] & p<1\end{cases} \tag{5.96}$$

By this equation [135, 127], one accounts for an infinite number of Bethe equations and substitutes the computation of a thermodynamic number of Bethe roots with the self-consistent determination of non-extensive number of effective excitations.

The *ground state* of the sine-Gordon model is realized as the unique state which has all the roots on the real axis, quantized with all consecutive half-integer quantization numbers. In the scaling limit, excited states are seen to have positive energy with respect to this state and to be completely specified by the quantum numbers of the holes and of the complex roots, whose combinations correspond to the IR excitations. The rapidities corresponding to these

objects are determined according to (5.56), and self-consistently determine the source term in (5.92), hence the counting function itself.

As seen in section 5.1.2, the total spin of an excited state above the antiferromagnetic ground state is, in field-theoretical language, the topological charge Q of the state itself. One can paraphrase the relation (1.53) for the total spin in terms of the excitations defined above as:

$$Q = 2S = N_H - M_C - 2N_S - (1 + \mathrm{sgn}(p-1))M_W \tag{5.97}$$

In general [135], it can be stated that in the repulsive regime, each hole in the source terms carries a unit $U(1)$ charge, which in the language of spin chain corresponds to a unit spin, i.e., to a missing creation operator. Such a charge is lowered by one for every close root and by two for every wide root. On the other hand, in the attractive regime, wide roots correspond to independent excitations, not carrying any $U(1)$ charge. We then expect that the wide roots correspond to creation operators, therefore lowering the spin by one, only in the repulsive regime $p > 1$. Conversely, in the attractive regime $p < 1$, wide roots enter the expression of a state only through their effect on the other roots and the determination of a counting function.

With respect to the infrared description of the spectrum, it is known [127, 128, 129] that

- *soliton* and *antisolitons* correspond to holes in the Fermi sea, quantized with half-integers.
- the solitons *polarization* states are described by arrays of the first kind, having common real part and containing in any case exactly one pair of close roots. The position of the roots for asymptotically large volumes [125] can be determined with exponential precision in the size and is reported here below. Arrays can be either:
  - odd degenerate

    $$\theta_0 = \theta + i\pi\frac{p+1}{2}\,, \qquad \theta_k = \theta \pm i\pi\left(\frac{1-p}{2} - kp\right) \qquad k = 0, 1, \ldots, \left\lfloor \frac{1}{2p} \right\rfloor \tag{5.98}$$

    with real θ;
  - even

    $$\theta_k = \theta \pm i\pi\left(\frac{1}{2} - kp\right) \qquad k = 0, 1, \ldots, \left\lfloor \frac{1}{2p} \right\rfloor \tag{5.99}$$

- the *breather* degrees of freedom, when $p > 1$, are described by arrays of the second kind, containing wide pairs only. These are, for $m\,L \to \infty$:
  - odd degenerate

    $$\theta_0 = \theta + i\pi\frac{p+1}{2}\,, \qquad \theta_k = \theta \pm i\pi\left(\frac{1-p}{2} - kp\right) \qquad k = 0, 1, \ldots, s \tag{5.100}$$

  - even

    $$\theta_k = \theta \pm i\pi\left(\frac{1}{2} - kp\right) \qquad k = 0, 1, \ldots, s \tag{5.101}$$

  with $0 \leq s \leq \frac{1}{2p} - 1$ and real θ. In particular, they describe the $2s+1$ and the $2s+2$ breather, respectively.

For simplicity, we will consider in the following only states in which the number of special objects is null and the counting function is monotonic on the real axis, which is the case for sufficiently large values of the size. With proper modification, the treatment can be in principle extended to account also for non monotonic counting functions, but this appears to be more cumbersome and will not be reported here.

The function $Z(\theta)$ is suitable for numeric computation and can be determined in a time of the order of minutes for the simplest root configurations. For a fairly larger amount of time, one can determine the counting function on a suitable grid on the complex plane, even if only its knowledge on three contours surrounding the real axis and the complex roots will be needed in the following.

5.2 Finite volume correlation functions of the sine-Gordon theory

The goal of this section is that of deriving an exact expression for the generating function of correlation functions in finite volume, including exponential corrections in the size. We will show that the finite-volume vacuum expectation value of the generating function is given by the expansion:

$$\left\langle e^{i\alpha(\phi(x)-\phi(0))}\right\rangle_L = \mathcal{G}_\alpha(L)\sum_\Psi e^{-i(\mathcal{E}(\Psi)-\mathcal{E}_0)t+i\mathcal{P}(\Psi)x}\mathcal{A}_L(\Psi) \tag{5.102}$$

where the brackets on the left-hand side denote the vacuum expectation value in finite volume L and on the right there appear the finite–size energies and momenta of the exact eigenstates Ψ and the generating functions of the finite-size form–factors.

The framework will be that of lattice integrable regularizations of the SG field theory and in particular of the one proposed in [124]. The expression derived will be written in terms of the counting function which solve the Destri-De Vega nonlinear integral equation (NLIE) as an expansion over the basis of the eigenstates of the theory: despite being in principle numerically computable, it assumes the knowledge of the solutions of the NLIE relative to the system with given coupling in a given volume, that is only partly available at the moment. We will moreover take advantage of the algebraic Bethe ansatz results relative to the one-dimensional (inhomogeneous) Heisenberg magnet and in particular of the computation of the matrix elements of the magnetization operator in the limit [137, 138] in which the number of sites goes to infinity.

By means of algebraic Bethe ansatz, a determinant representation for the generating function has been the goal of [139], a work based on the lattice sine-Gordon regularization proposed in [140]. We believe that the subsequent advances on quantum spin chains (such as [13]) allow more explicit results. In connection with the inhomogeneous XXZ spin chain, the one-point functions of primary fields and their descendants in the sine-Gordon model have been analysed in the framework of the recently explored fermionic structure of the model in [141, 142], while the computation of form factors by separation-of-variables has been tackled in [143] recently.

5.2.1 The generating function

A convenient method for writing connected correlation functions is by differentiation of a generating function. In particular, we are interested in the expectation value on the vacuum

of the operator:

$$\left\langle e^{-2i\pi\omega/\beta(\phi(x)-\phi(0))}\right\rangle_L \tag{5.103}$$

in which ω is here a real number and the subscript L stands for the size of the system. The field in the exponent is proportional to the fraction of topological charge in the interval $[0,x]$ (see [124]) and is realized on the lattice by a string of operators acting on the local spin Hilbert spaces as:

$$(\phi(x)-\phi(0)) = \frac{\beta}{2\pi}\sum_{l=1}^{2m}\sigma_l^z \tag{5.104}$$

with σ^z the usual Pauli matrix. A similar operator, the sum over projectors on the spin-up state, reads:

$$\mathcal{Q}_{2m+1} = \frac{1}{2}\sum_{l=1}^{2m}(1-\sigma_l^z) \tag{5.105}$$

A convenient representation of the exponential of the operator (5.105) was provided in [137] in terms of the transfer matrices $\hat{\tau}_0, \hat{\tau}_\omega$ of two spin chains: one corresponding to the actual physical system, and the other to an analogous system in which a twist in the boundary condition ω had been introduced the results presented in section 1.3.2, the solution of the inverse scattering problem for arbitrary inhomogeneities ξ_l $l = 1,\ldots,2N$. The expression for the magnetization operator (1.68) allows to write down [137] the generating function in the inhomogeneous chain as:

$$e^{-i\omega\sum_{l=0}^m\sigma_l^z} = \prod_{l=0}^m e^{-i\omega}\hat{\tau}_\omega\left((-1)^l\Lambda - i\gamma/2\right)\left(\prod_{l=1}^m\hat{\tau}_0\left((-1)^l\Lambda - i\gamma/2\right)\right)^{-1} \tag{5.106}$$

We will associate here the set of rapidities $\{\mu\}$ to the ground state and introduce a complete set of eigenstates of the twisted transfer matrix on the right of this operator. The transfer matrices act diagonally on the respective eigenstates, so that the expression obtained is:

$$\frac{\left\langle\{\mu\}\left|e^{i\omega\sum_j^m\sigma_j^z}\right|\{\mu\}\right\rangle_L}{\left\langle\{\mu\}\middle|\{\mu\}\right\rangle_L} = \sum_{\{\lambda\}_\omega}\mathcal{A}(\{\lambda\})\prod_{l=1}^m e^{-i\omega}\frac{\tau_\omega((-1)^l\Lambda - i\gamma/2|\{\lambda\}_\omega)}{\tau_0((-1)^l\Lambda - i\gamma/2|\{\mu\})} \tag{5.107}$$

with

$$\mathcal{A}(\{\lambda\}_\omega) = \frac{|\langle\{\lambda\}_\omega|\{\mu\}\rangle_L|^2}{\langle\{\mu\}|\{\mu\}\rangle_L\langle\{\lambda\}|\{\lambda\}\rangle_L} \tag{5.108}$$

The product on the right hand side yields, in the scaling limit, the phase:

$$\prod_{l=1}^m e^{-i\omega}\frac{\tau_\omega((-1)^l\Lambda - i\gamma/2|\{\lambda\}_\omega)}{\tau_0((-1)^l\Lambda - i\gamma/2|\{\mu\})} \to e^{-ix(\mathcal{P}(\{\lambda\}_\omega)-\mathcal{P}(\{\mu\}))} \tag{5.109}$$

which has been seen in section 5.1.2.

What written above is then a formal decomposition of the generating function of connected correlation functions. Its derivatives with respect to the twist ω provide a form factor expansion familiar in the framework of field theory. For simplicity, we retain the form (5.107) and refer to it as amplitude expansion.

Once properly normalized, the overlap of any twisted state on the ground state is a complex number with modulus between zero and one; nonetheless, it will be convenient for

comparison with the field theory formalism to compute the scaling limit with respect to a reference amplitude:

$$\mathcal{A}(\{\lambda\}_\omega,\{\mu\}_\omega,\{\mu\}) = \frac{\langle\{\mu\}_\omega|\{\mu\}_\omega\rangle_L}{|\langle\{\mu\}_\omega|\{\mu\}\rangle_L|^2}\frac{|\langle\{\lambda\}_\omega|\{\mu\}\rangle_L|^2}{\langle\{\lambda\}_\omega|\{\lambda\}_\omega\rangle_L} \tag{5.110}$$

Hence, we rewrite the above generating function in finite volume in terms of "normalized" amplitudes as:

$$\left\langle e^{2i\omega Q_m}\right\rangle_L = \mathcal{A}_0\left(1+\sum_{excited\ \{\lambda\}_\omega} e^{-ix\mathcal{P}_{\{\lambda\}}}\mathcal{A}(\{\lambda\}_\omega,\{\mu\}_\omega,\{\mu\})\right) \quad , \tag{5.111}$$

where

$$\mathcal{A}_0 = \frac{|\langle\{\mu\}_\omega|\{\mu\}\rangle_L|^2}{\langle\{\mu\}|\{\mu\}\rangle_L\,\langle\{\mu\}_\omega|\{\mu\}_\omega\rangle_L} \tag{5.112}$$

The amplitudes in (5.107) will be the object of our concern in the next sections. The factor $\mathcal{A}_0$ represents an overall normalization of the exponential operator. Its physical meaning in the scaling limit can be argued by considering very large values of x, in which all the terms in the expansion become rapidly oscillating except the first one[1]: we therefore expect it to correspond to the (squared) vacuum expectation value of the operator $e^{2\pi i\omega/\beta\phi(x)}$ in finite size.

The intermediate states are defined by the root structure which is encoded in the source term of (5.92), i.e., by the number of holes, close, ... roots and by their quantization numbers. The question about the completeness of the presently known solutions of the nonlinear integral equation is still open, to our knowledge.

5.2.2 The result

Suppose the ket $|0\rangle_L$ to be the (finite volume) vacuum, which corresponds to the state in which all the roots lie on the real axis and are quantized by half-integers, without holes. The set of integers $\{\bar{I}\}$ defines instead an the excited state in the twisted system. The sine-Gordon sector [127, 128, 129] is reproduced by the configuration of roots having $2S+\delta+M_{sc}\in 2\mathbf{Z}$ and we will consider, for definiteness, half-integer quantization numbers for the rapidities, i.e. $\delta=0$. Hence, in the following, the number of self-conjugated roots is required to be even. To have a non vanishing matrix element, it is moreover necessary that the total number of roots in the excited state is the same as that of the ground state.

The amplitudes have the property:

$$\mathcal{A}(\{\bar{I}\},x,t) = e^{-i\mathcal{P}(\{\lambda\})x+i(\mathcal{E}(\{\lambda\})-\mathcal{E}_0)t}\mathcal{A}(\{\bar{I}\},0,0) \tag{5.113}$$

where $\mathcal{P}(\{\lambda\})$ is the total dressed momentum of the state $\Psi(\{\lambda\})$ and $\mathcal{E}(\{\lambda\})$ its energy, while $\mathcal{E}_0$ refers to the energy of the ground state. Then,

$$\begin{aligned}\mathcal{P} &= \sum_j^{N_h} m\sinh h_j - \sum_j^{N_s}\left(m\sinh\,y_s^+ + m\sinh\,y_s^-\right) - \sum_j^{M_c} m\sinh c_j - \sum_j^{M_w} m\sinh w_j \\ &\quad -\frac{1}{\pi}\int d\theta'\,\cosh\theta'\Im\log(1+e^{iZ(\theta'^+)})\end{aligned} \tag{5.114}$$

[1] upon subtraction of the bulk momentum 2ω.

$$\mathcal{E}-\mathcal{E}_{bulk} = \sum_j^{N_h} m\cosh h_j - 2\sum_j^{N_s} m\cosh\, y_s - \sum_j^{Mc} m\cosh c_j - \sum_j^{M_w} m\cosh w_j$$
$$-\frac{1}{\pi}\int d\theta'\, \sinh\theta'^{+}\Im\log(1+e^{iZ(\theta'^{+})}) \tag{5.115}$$

for which explicit computations can be found in [136, 144, 127, 128, 129, 134, 135]. The time shift phase comes from applying the double-row transfer matrix along the vertical direction.

Here below and in the following, we will denote the indexes relative to the holes of the excited states by h and the ones relative to the complex roots, generically, by a c. This shouldn't generate confusion with the "close" roots, as notation will be clear from the context. Moreover, to shorten notations, we write the signs $\{c_j\}$, with the convention that $c_{holes} = 1$, $c_{complex\,roots} = -1$. We define hereby the functions:

$$\varphi_\rho(x,y) = \frac{mL\,\sinh\frac{\gamma}{\pi}(x-y)}{Z_\rho(x)-Z_\rho(y)} \tag{5.116}$$

$$\begin{aligned}
\mathcal{L}_0^\sigma(x) &= \int du\Delta(x-u)\log(1+e^{i\sigma Z_\mu(u^\sigma)}) \\
\mathcal{L}_\lambda^\sigma(x) &= \int du\Delta(x-u)\log(1+e^{i\sigma Z_\lambda(u^\sigma)}) \\
\mathcal{L}^\sigma(x) &= \int du\Delta(x-u)\log\frac{1+e^{i\sigma Z(u^\sigma)}}{1+e^{i\sigma Z_0(u^\sigma)}}
\end{aligned} \tag{5.117}$$

With this notation, the expression for the amplitude defined by the twisted excited state $\{\lambda\}$, evaluated in the origin, is:

$$|\mathcal{A}(\{I\})|^2 = a_\omega \frac{\mathcal{S}\,\Phi\,\mathcal{D}}{\cosh^2\Sigma}\prod_c \mathcal{C}(\lambda_c)\prod_h \mathcal{H}(\lambda_h)\mathcal{R} \tag{5.118}$$

in which there appear the quantities:

$$\mathcal{R} = \left|\frac{\det\left[1-\hat{W}_{\mu,\lambda}\right]\det\left[1+(A_+-1)\,\hat{G}_{-\omega}\right]}{\det\left[1-\hat{W}_\mu\right]}\frac{\det\left[1-\hat{W}_{\lambda,\mu}\right]\det\left[1+\left(A_+^{-1}-1\right)\hat{G}_\omega\right]}{\det\left[1-\hat{W}_\lambda\right]}\right| \tag{5.119}$$

$$\mathcal{D} = \frac{\prod_{c\neq c'}\sinh\frac{\gamma}{\pi}(\lambda_c-\lambda_{c'})\prod_{h\neq h'}\sinh\frac{\gamma}{\pi}(\lambda_h-\lambda_{h'})}{\prod_{ch}\sinh\frac{\gamma}{\pi}(\lambda_c-\lambda_h)\sinh\frac{\gamma}{\pi}(\lambda_c-\lambda_h)} \tag{5.120}$$

$$a_\omega = \left|\frac{\det\left[1+\hat{K}_\omega\right]}{\det\left[1+\hat{K}\right]}\right|^2 \tag{5.121}$$

$$\begin{aligned}
\mathcal{C}(\lambda_c) &= \frac{\pi}{\gamma}\frac{\cos\frac{Z_0(\lambda_c)}{2}}{Z'(\lambda_c)}\exp\Big[-\frac{2\gamma}{\pi^2}\int du\Im\mathcal{L}_+(u)\coth\frac{\gamma}{\pi}(\lambda_c-u) \\
&\quad +2\int du\left(\sum_h G(\lambda_h-u)-\sum_c G(\lambda_c-u)-\sum_w G_{II}(\lambda_w-u)\right)\log\sinh\frac{\gamma}{\pi}(\lambda_c-u)\Big] \\
\mathcal{H}(\lambda_h) &= \frac{\pi}{\gamma}\frac{\cos\frac{Z_0(\lambda_h)}{2}}{Z'(\lambda_h)}\exp\Big[-2\,\mathrm{sgn}(1-p)M_w \\
&\quad +2\int\frac{du}{\pi}\Im\left(\mathcal{L}_0^+(u)\partial_x\log\varphi_0(\lambda_h,u^+)-\mathcal{L}_\lambda^+(u)\partial_x\log\varphi_\lambda(\lambda_h,u^+)\right) \\
&\quad -2\int du\left(\sum_h G(\lambda_h-u)-\sum_c G(\lambda_c-u)-\sum_w G_{II}(\lambda_w-u)\right)\log\varphi_\lambda(\lambda_h,u^-)\Big] \qquad (5.122)
\end{aligned}$$

$$\begin{aligned}
\Phi &= \exp\Big[\sum_{j,k=h,c}\int dx c_j G(\lambda_j-x)\int dy c_k G(\lambda_k-y)\log\varphi(x,y)-M_W^2 \\
&\quad -\sum_{\sigma,\sigma'=\pm}\frac{\sigma\sigma'}{(2\pi)^2}\int dx\mathcal{L}^{\sigma'}(x)\int dy\big[\mathcal{L}_\lambda^\sigma(x)\partial_{x,y}^2\log\varphi(x^\sigma,y^{\sigma'})-\mathcal{L}_0^\sigma(x)\partial_{x,y}^2\log\varphi_0(x^\sigma,y^{\sigma'})\big] \\
&\quad +\sum_\alpha c_\alpha\int\frac{dx}{\pi}G(\lambda_\alpha-x)\int dy\Im\left[\mathcal{L}_\lambda^+(x)\partial_y\log\varphi(x,y^+)-\mathcal{L}_0^+(x)\partial_y\log\varphi_0(x,y^+)\right]\Big]
\end{aligned} \qquad (5.123)$$

$$\begin{aligned}
\mathcal{S} &= \exp\Big[\frac{\gamma^2}{\pi^2}\sum_{\sigma\sigma'=\pm}\frac{\sigma\sigma'}{(2\pi i)^2}\int dx\int dy\mathcal{L}_\sigma(x)\mathcal{L}_{\sigma'}(y)\frac{1}{\sinh^2\frac{\gamma}{\pi}(x^\sigma-y^{\sigma'}-i\pi)} \\
&\quad +\int dx\int dy\hat\Delta(x)\hat\Delta(y)\log\sinh\frac{\gamma}{\pi}(y-x-i\pi) \\
&\quad +\frac{\gamma}{\pi}\sum_{\sigma=\pm}\frac{\sigma}{2\pi i}\int dx\int dy\hat\Delta(x)\mathcal{L}_\sigma(y)\frac{\sinh\frac{\gamma}{\pi}(y^\sigma-x)}{\sinh\frac{\gamma}{\pi}(y^\sigma-x-i\pi)\sinh\frac{\gamma}{\pi}(y^\sigma-x+i\pi)}\Big]
\end{aligned} \qquad (5.124)$$

We have made use of the notation $\hat\Delta$ to denote:

$$\int du\hat\Delta(u)f(u)=\sum_j{}'c_jf(\lambda_j)-\sum_j c_j\int duG(\lambda_j-u)f(u) \qquad (5.125)$$

where the sum is over all the sources appearing in (5.92) and the prime excludes the wide roots from the sum if $p<1$.

$$\Sigma=\int\frac{dx}{2\pi}x\left(Z'(x)-Z_0'(x)\right)\left(\sum_{\sigma=\pm}\left(\frac{1}{1+e^{\varepsilon_\sigma(x|\{\lambda\})}}-\frac{1}{1+e^{\varepsilon_\sigma(x)}}\right)-1\right)-\sum_j c_j\lambda_j \qquad (5.126)$$

The soliton and antisoliton pseudoenergies [145] satisfy the integral equations:

$$\begin{aligned}
\varepsilon_+(\theta) &= mL\cosh\theta-\int d\theta' G_+(\theta-\theta')\log(1+e^{\varepsilon_+}(\theta'))+\int d\theta' G_-(\theta-\theta')\log(1+e^{\varepsilon_-(\theta')}) \\
\varepsilon_+(\theta|\{\lambda\}) &= mL\cosh\theta+ig(\theta+i\pi/2|\{\lambda\})-\sum_{\sigma=\pm}\int d\theta' G_\pm(\theta-\theta')\log(1+e^{\varepsilon_\sigma(\theta'|\{\lambda\})})
\end{aligned} \qquad (5.127)$$

and

$$\varepsilon_-(\theta) = \bar{\varepsilon}_+(\theta) \qquad G_+(\theta) = G(\theta) \qquad G_-(\theta) = G(\theta + i\pi^-) \tag{5.128}$$

With these functions, one can express the integral operators appearing in the overlap determinant. They depend either on two complex variables w, v and on two species indexes $\sigma, \sigma' = \pm$ ("bulk" parameters) or on the rapidities that define the excitations. They read:

$$\begin{aligned}
W_{\lambda,\mu}^{\sigma,\sigma'}(w,v) &= \frac{1}{2\pi}\frac{A(w+i\sigma\pi/2)}{1+e^{\varepsilon_\sigma(w)}}\left(G_{-\omega}\left(w-v+i\frac{\sigma-\sigma'}{2}\pi^-\right)+F_{-\omega}^{\sigma,\sigma'}(w,v)\right)\\
W_{\lambda,\mu}(\lambda_c,v)^{\sigma'} &= \frac{ResA(\lambda_c)}{1+e^{iZ_0(\lambda_c)}}\left(G_{-\omega}\left(\lambda_c-v-i\frac{\sigma'\pi}{2}\right)+F_{-\omega}^{\sigma'}(\lambda_c,v)\right)\\
W_{\lambda,\mu}(w,\lambda_c)^{\sigma} &= \frac{1}{2\pi}\frac{A(w+i\sigma\pi/2)}{1+e^{\varepsilon_\sigma(w)}}\left(G_{-\omega}\left(w-\lambda_c+i\frac{\sigma}{2}\pi\right)+F_{-\omega}^{\sigma}(w,\lambda_c)\right)\\
W_{\lambda,\mu}(\lambda_c,\lambda_{c'}) &= \frac{ResA(\lambda_c)}{1+e^{iZ_0(\lambda_c)}}\left(G_{-\omega}(\lambda_c-\lambda_{c'})+F_{-\omega}(\lambda_c,\lambda_{c'})\right)
\end{aligned} \tag{5.129}$$

$$\begin{aligned}
W_{\mu,\lambda}^{\sigma,\sigma'}(w,v) &= \frac{1}{2\pi}\frac{A^{-1}(w+i\sigma\pi/2)}{1+e^{\varepsilon(w|\{\lambda\})}}\left(G_{\omega}\left(w-v+i\frac{\sigma-\sigma'}{2}\pi^-\right)+F_{\omega}^{\sigma,\sigma'}(w,v)\right)\\
W_{\mu,\lambda}^{\sigma'}(\lambda_h,v) &= -\frac{A^{-1}(\lambda_h)}{Z'(\lambda_h)}\left(G_{\omega}\left(\lambda_h-v-i\frac{\sigma'\pi}{2}\right)+F_{\omega}^{\sigma'}(\lambda_h,v)\right)\\
W_{\mu,\lambda}^{\sigma}(w,\lambda_h) &= \frac{1}{2\pi}\frac{A^{-1}(w+i\sigma\pi/2)}{1+e^{\varepsilon(w|\{\lambda\})}}\left(G_{\omega}\left(w-\lambda_h+i\frac{\sigma}{2}\pi\right)+F_{\omega}^{\sigma}(w,\lambda_h)\right)\\
W_{\mu,\lambda}(\lambda_h,\lambda_{h'}) &= -\frac{A^{-1}(\lambda_h)}{Z'(\lambda_h)}\left(G_{\omega}(\lambda_h-\lambda_{h'})+F_{\omega}(\lambda_h,\lambda_{h'})\right)
\end{aligned} \tag{5.130}$$

and

$$\begin{aligned}
A(w) &= \exp\Big[i\frac{\gamma}{\pi}\sum_{\sigma=\pm}\sigma\int\frac{dx}{2\pi}\left(\coth\frac{\gamma}{\pi}(w-x^\sigma-i\pi)-\coth\frac{\gamma}{\pi}(w-x^\sigma)\right)\mathcal{L}^\sigma(x)\\
&\quad +i\int dx\hat{\Delta}(x)\log\frac{\sinh\frac{\gamma}{\pi}(x-w-i\pi)}{\sinh\frac{\gamma}{\pi}(x-w)}\Big]
\end{aligned} \tag{5.131}$$

and in particular:

$$A(w+i\pi/2) = A_+(w) = e^{-i(Z(w)-Z_0(w))-2i\omega} \tag{5.132}$$

The determinants are of the Fredholm type, integrals are performed on the real axis and the the species indexes and the excitations variables are summed over as well. In facts, complex roots of the bra state must be explicitly summed over as well as holes of the ket, if any, subtracted. Strictly speaking, the above representation is valid only for $p > 1/2$. However, it will be clear how to extend it for smaller values once the derivation is made in section 5.2.4. The source function needs, for some configurations, to be evaluated in regions in which the imaginary part of the argument exceeds $\min(p,1)\pi$: it is therefore necessary to use the second determination.

The integral operator G_ω is defined as:

$$G_\omega(w) = \int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{i\,k\,w}\frac{\sinh\left[\left(\frac{\pi}{2}-\gamma\right)k-i\omega\right]}{e^{-i\omega}\sinh\frac{\pi k}{2}+\sinh\left[\left(\frac{\pi}{2}-\gamma\right)k-i\omega\right]} \tag{5.133}$$

which reduces to (5.83) for $\omega \to 0$. Moreover,

$$F_{\omega}^{\sigma,\sigma'}(w,v) = \sum_{n=1}^{\infty} \int dx_1 \dots dx_n G_{\omega}\left(w - x_1 + i\frac{\sigma-1}{2}\pi^-\right)(1 - A_+(x_1))\, G_{\omega}(x_1 - x_2)\dots \\ \dots (1 - A_+(x_n))\, G_{\omega}\left(x_n - v + i\frac{1-\sigma'}{2}\pi^-\right) \tag{5.134}$$

with and obvious extension to the case where one or both rapidities in the argument appear in the source.

For evaluating the overall normalization amplitude $\mathcal{A}_0$ one also needs the factor a_ω in (5.118), with kernel:

$$K_{\omega}(x) = \coth\frac{\gamma}{\pi}(x - i\pi) - e^{2i\omega}\coth\frac{\gamma}{\pi}(x + i\pi) \tag{5.135}$$

and x on the real axis. The latter determinants are not of trace class and we have not found a way of evaluating them so far.

For what the norm determinants are concerned, their expression can be written as:

$$\det\left[1 - \hat{W}_x\right] \tag{5.136}$$

where x stands for one of the two states and:

$$\begin{aligned}
W_{\mu}^{\sigma,\sigma'}(w,v) &= \frac{1}{2\pi}\frac{1}{1+e^{\varepsilon_\sigma(w)}}G\left(w - v + i\frac{\sigma-\sigma'}{2}\pi^-\right) \\
W_{\lambda}^{\sigma,\sigma'}(w,v) &= \frac{1}{2\pi}\frac{1}{1+e^{\varepsilon_\sigma(w|\{\lambda\})}}G\left(w - v + i\frac{\sigma-\sigma'}{2}\pi^-\right) \\
W_{\lambda}^{\sigma'}(\lambda_h,v) &= -\frac{1}{Z'(\lambda_h)}G\left(\lambda_h - v - i\frac{\sigma'\pi}{2}\right) \\
W_{\lambda}^{\sigma}(w,\lambda_h) &= \frac{1}{2\pi}\frac{1}{1+e^{\varepsilon_\sigma(w|\{\lambda\})}}G\left(w - \lambda_h + i\frac{\sigma}{2}\pi\right) \\
W_{\lambda}(\lambda_h,\lambda_{h'}) &= -\frac{1}{Z'(\lambda_h)}G(\lambda_h - \lambda_{h'})
\end{aligned} \tag{5.137}$$

where the "holes" terms are present for a generic excited state.

An interpretation in terms of pseudoparticles is therefore possible: the finite-size vacuum can be written in terms of the fundamental excitations (solitons and antisolitons) of the infrared theory, which occupy the available levels according to a filling fraction containing the vacuum and excited pseudoenergies. Excitations constructed upon such a vacuum interact both among them and with the background pseudoparticles and the matrix elements of the operators show such features of the theory.

5.2.3 Scalar products and norms

We need to perform the computation of the scalar products of the ground state with a generic "twisted" state. The computation of scalar products in the framework of algebraic Bethe ansatz has a long history. We use below the formulas of section 1.3.2. From eq. (1.65), we can extract both the overlaps and the norms of the states after some manipulation, in which we make explicit use that the rapidities $\{\lambda\}$ also satisfy (5.53).

Two alternative expressions, which are suitable for the scaling limit, can be provided for the overlap. We refer the reader to [137], where the determinant of the overlap matrix was written as a Fredholm determinant on a contour. With some variation of their method, in which use of the Bethe equations and of the definition (5.51) is explicitly made, the scalar product (1.65) can be written in a form which is more suitable for subsequent manipulation.

To compute the determinant, we multiply the matrix H by a conveniently defined matrix M and its inverse. In the case of a twisted state defined by the roots $\{\lambda\}$ and the state defined by the roots $\{\mu\}$, we consider the matrix

$$M_{j,k} = \frac{\cosh(\mu_j - \lambda_k) \prod_{l \neq k} \sinh(\mu_j - \lambda_l)}{\prod_{l \neq j} \sinh(\mu_j - \mu_l)} \tag{5.138}$$

whose elements are $i\pi$-antiperiodic functions of the rapidity μ_j. Its determinant is

$$\det M = \prod_{j<k} \frac{\sinh(\lambda_j - \lambda_k)}{\sinh(\mu_j - \mu_k)} \cosh(\sum_l \lambda_l - \sum_l \mu_l) \tag{5.139}$$

Then the matrix product $\tilde{H}_{jn} M_{nk}$ can be computed by considering the integral

$$\oint \frac{dw}{2\pi i} \frac{\sinh(-i\gamma)}{\sinh(w - \lambda_j) \sinh(w - \lambda_j \pm i\gamma)} \frac{\cosh(w - \lambda_k) \prod_{l \neq k} \sinh(w - \lambda_l)}{\prod_l \sinh(w - \mu_l)} \tag{5.140}$$

which is vanishing when the contour of integration surrounds the real axis and the strip $[-\pi/2, \pi/2]$ along the imaginary axis.

Then the result of the matrix multiplication is

$$\begin{aligned} [H^\omega \cdot M]_{a,b} &= (-e^{-2i\omega}) d(\lambda_a) \left(1 + B_\lambda^\omega(\mu_a)\right) \prod_l \sinh(\lambda_l - \mu_a + i\gamma) \frac{\prod_{l \neq a} \sinh(\mu_a - \mu_l)}{\prod_l \sinh(\mu_a - \lambda_l)} \\ &\quad \Big\{ \delta_{ab} - \frac{\prod_l \sinh(\mu_a - \lambda_l)}{1 + B_\lambda^\omega(\mu_a)} \frac{1}{\prod_{l \neq a} \sinh(\mu_a - \mu_l)} \frac{\sinh(\mu_a - \mu_l - i\gamma)}{\sinh(\mu_a - \lambda_l - i\gamma)} \\ &\quad \left(\coth(\mu_a - \mu_b - i\gamma) - e^{2i\omega} \coth(\mu_a - \mu_b + i\gamma) \right) \Big\} \end{aligned} \tag{5.141}$$

from which the Fredholm determinant in the limit in which the size of the matrices goes to infinity can be recovered.

In the following expressions, the quantity ω denotes the relative twist of the state $|\Psi_\omega\{\lambda\}\rangle$.

$$\begin{aligned} \langle \psi(\{\lambda_j\}) | \psi(\{\mu\}) \rangle &= \frac{e^{-2i\omega M} \prod_j d(\lambda_j) d(\mu_j) \left(1 + B_\mu(\lambda_j)\right)}{\cosh(\sum \lambda_l - \sum \mu_l)} \prod_{j,k} \frac{\sinh(\mu_j - \lambda_k + i\gamma)}{\sinh(\mu_j - \lambda_k)} \det\left(1 - \hat{U}^{-\omega}\right) \\ &= \frac{\prod_j d(\lambda_j) d(\mu_j) \left(1 + B_\lambda(\mu_j)\right)}{\cosh(\sum \lambda_l - \sum \mu_l)} \prod_{j,k} \frac{\sinh(\lambda_j - \mu_k + i\gamma)}{\sinh(\mu_j - \lambda_k)} \det\left(1 - \hat{U}^{\omega}\right) \end{aligned} \tag{5.142}$$

with the matrix

$$U_{j,k}^{-\omega} = \frac{K_{-\omega}(\lambda_j - \lambda_k)}{1 + B_\mu(\lambda_j)} \frac{\prod_l \sinh(\lambda_j - \mu_l)}{\prod_{l \neq j} \sinh(\lambda_j - \lambda_l)} \prod_l \frac{\sinh(\lambda_j - \lambda_l - i\gamma)}{\sinh(\lambda_j - \mu_l - i\gamma)} \tag{5.143}$$

$$U_{j,k}^{\omega} = \frac{K_{\omega}(\mu_j - \mu_k)}{1 + B_\lambda(\mu_j)} \frac{\prod_l \sinh(\mu_j - \lambda_l)}{\prod_{l \neq j} \sinh(\mu_j - \mu_l)} \prod_l \frac{\sinh(\mu_j - \mu_l - i\gamma)}{\sinh(\mu_j - \lambda_l - i\gamma)} \tag{5.144}$$

and the function (5.135).

For the computation of norms, one considers the limit $\{\lambda\}, \{\mu\} \to \{\nu\}$, for which the matrix above becomes simply:

$$U^{\omega_\nu}_{j,k} = \frac{K(\nu_j - \nu_k)}{1 + B^{\omega_\nu}_\nu(\nu_j)} \tag{5.145}$$

For the remaining part of the section, we shall be using rescaled rapidity variables. Moreover, unless otherwise specified, we shall consider the state μ to be the ground state of the (untwisted) inhomogeneous chain, while the state $\{\lambda\}$ is considered as having a twist.

As a preliminary step, one observes that by applying the definition of counting function and a representation of the cosine as an infinite product:

$$1 + B^\omega_\mu(x) = 2e^{-i/2\, Z_\mu(x)} \prod_{k=-\infty}^{\infty} \left(1 - \frac{Z_\mu(x)}{2\pi(k-1/2)}\right) \tag{5.146}$$

from which, considering the ground state with $2M$ roots $\{\mu\}$, having labels ranging on half-integers between $-M+1/2$ and $M-1/2$, we have:

$$1 + B^\omega_\mu(x) = 2e^{-i/2\, Z_\mu(x)} \prod_{I=-M+1/2}^{M-1/2} \left(\frac{Z_\mu(\mu_I) - Z_\mu(x)}{Z_\mu(\mu_I)}\right) \frac{\Gamma(M+\frac{1}{2})^2}{\Gamma(M+\frac{1}{2}-\frac{Z_\mu}{2\pi})\Gamma(M+\frac{1}{2}+\frac{Z_\mu}{2\pi})} \tag{5.147}$$

The last ratio tends to unity in the limit in which the number of roots goes to infinity and will not be rewritten in the following. The case in which there is a finite number of excitations yields the same result, if one considers in the product above the set $\{\tilde{\mu}\}$ of all the real roots of (5.60).

We underline that for the states in which the Bethe roots are quantized with integers, an analogous product representation for the sine allows to rewrite the factor $1 - B^\omega_\mu(x)$ in terms of differences of counting functions evaluated at distinct values of the argument. All the subsequent scheme of calculation can be straightforwardly adjusted.

Having established this fact, we consider the state $\{\lambda\}$ to be excited and the state $\{\mu\}$ to be the vacuum, identified hereby by the subscript 0. We are moreover interested in the normalized matrix elements, so we divide the overlap by the norm of the two states.

Let us multiply and divide by the holes and the complex roots, in order to obtain expressions in which all and only the *real solutions* appear. This is convenient in that we can consider the ratio between each hyperbolic sine appearing in the denominator of the expressions (5.142) and the differences of the counting function computed at the points in the argument of the sine, as arising from the product representation (5.147). Following [137] this defines the functions (5.116).

After illustrating the general procedure, it is simpler to consider two additional states, whose rapidities we label by $\{\rho\}$, $\{\nu\}$: at the end of the computations, we will send $\{\rho\} \to \{\lambda\}$ and $\{\nu\} \to \{\mu\}$ and show that the poles arising from the factors of the kind (5.60) are cancelled by the zeros of the hyperbolic sines in the expression for the scalar product. In order to obtain

a product involving only the real solutions, we consider the ratio:

$$\prod_{j,k}\frac{\sinh\frac{\gamma}{\pi}(\lambda_j-\rho_k)\sinh\frac{\gamma}{\pi}(\mu_j-\nu_k)}{\sinh\frac{\gamma}{\pi}(\lambda_j-\nu_k)\sinh\frac{\gamma}{\pi}(\mu_j-\rho_k)}=\frac{\prod_{cc'}\sinh\frac{\gamma}{\pi}(\lambda_c-\rho_{c'})\prod_{h,h'}\sinh\frac{\gamma}{\pi}(\lambda_h-\rho_{h'})}{\prod_{ch}\sinh\frac{\gamma}{\pi}(\lambda_c-\rho_h)\sinh\frac{\gamma}{\pi}(\rho_c-\lambda_h)}$$
$$\prod_{j,k}\frac{\sinh\frac{\gamma}{\pi}(\tilde{\lambda}_j-\tilde{\lambda}_k)\sinh\frac{\gamma}{\pi}(\mu_j-\nu_k)}{\sinh\frac{\gamma}{\pi}(\tilde{\lambda}_j-\mu_k)\sinh\frac{\gamma}{\pi}(\tilde{\rho}_j-\mu_k)}\prod_h\prod_j\frac{\sinh\frac{\gamma}{\pi}(\lambda_h-\mu_j)\sinh\frac{\gamma}{\pi}(\rho_h-\mu_j)}{\sinh\frac{\gamma}{\pi}(\lambda_k-\tilde{\rho}_j)\sinh\frac{\gamma}{\pi}(\rho_h-\tilde{\lambda}_j)}$$
$$\prod_c\prod_j\frac{\sinh\frac{\gamma}{\pi}(\lambda_c-\tilde{\rho}_j)\sinh\frac{\gamma}{\pi}(\rho_c-\tilde{\lambda}_j)}{\sinh\frac{\gamma}{\pi}(\lambda_c-\mu_j)\sinh\frac{\gamma}{\pi}(\rho_c-\mu_j)} \tag{5.148}$$

On the other hand, for what the factor involving the counting function is concerned, we can write:

$$\prod_j\frac{1+e^{iZ_0(\rho_j)}}{1+e^{iZ(\rho_j)}}=\prod_c\frac{1+e^{iZ_0(\rho_c)}}{1+e^{iZ(\rho_c)}}\prod_h\frac{1+e^{iZ_0(\rho_h)}}{1+e^{iZ(\rho_h)}}\prod_h\left(\frac{1+e^{iZ(\rho_h)}}{1+e^{iZ_0(\rho_h)}}\right)^2\prod_j\frac{1+e^{iZ_0(\tilde{\rho}_j)}}{1+e^{iZ(\tilde{\rho}_j)}} \tag{5.149}$$

By multiplying the first and the second product with the first ratio of (5.148) and taking the limit to coinciding states, we obtain the term (5.120), which already contains a finite number of rapidities, apart from a phase factor.

The factors in the first term that contain the same index for the hole, together with the third term of (5.148) and the third term of (5.149) yield:

$$\prod_h\mathcal{H}(\lambda_h)\quad,\qquad\mathcal{H}(\lambda(h))\;=\;\frac{1+e^{iZ_\mu(\lambda_h)}}{iZ'_\lambda(\lambda_h)}\prod_j\frac{\varphi_0(\lambda_h-\tilde{\mu}_j)^2}{\varphi(\lambda_h-\tilde{\lambda}_j)^2} \tag{5.150}$$

while the last of (5.149), accompanied by the corresponding product in the μ rapidities, with the second in (5.148) provide the factor:

$$\Phi=\frac{\varphi_0(\mu_j,\mu_k)\varphi(\tilde{\lambda}_j,\tilde{\lambda}_k)}{\varphi_0(\mu_j,\tilde{\lambda}_k)\varphi(\mu_j,\tilde{\lambda}_k)} \tag{5.151}$$

From all the previous expressions, one also obtains a phase factor containing sum over rapidities of the difference of the two counting functions, that will be of no relevance in the following.

The last product in (5.148) is already in a form suitable for the scaling limit; together with the part of the product in the first term that contains the same index for the close root, it may be rewritten as:

$$\prod_c\mathcal{C}(\lambda_c)\quad,\qquad\mathcal{C}(\lambda_C)=\frac{1+e^{iZ_\mu(\lambda_c)}}{iZ'_\lambda(\lambda_c)}\left(\prod_j\frac{\sinh\frac{\gamma}{\pi}(\lambda_c-\tilde{\lambda}_j)}{\sinh\frac{\gamma}{\pi}(\lambda_c-\tilde{\mu}_j)}\right)^2 \tag{5.152}$$

and constitutes a multiplicative contribution from complex roots.

According to our previous analysis, we write:

$$\frac{|\langle\Psi(\{\lambda\})|\Psi(\{\mu\})\rangle|^2}{\|\Psi(\{\mu\})\|^2\|\Psi(\{\lambda\})\|^2}=\mathcal{S}\;\mathcal{D}\;\Phi\;\prod_h\mathcal{H}(\lambda_h)\prod_c\mathcal{C}(\lambda_c)\left|\frac{\det\left(1-U^{\omega}_{\lambda,\mu}\right)\det\left(1-U^{-\omega}_{\mu,\lambda}\right)}{\det\left(1-U_\lambda\right)\det\left(1-U_\mu\right)}\right| \tag{5.153}$$

Where the definitions (5.120) and

$$\mathcal{S} = \prod_{j,k} \frac{\sinh(\mu_j - \lambda_k - i\gamma)\sinh(\lambda_j - \mu_k - i\gamma)}{\sinh(\mu_j - \mu_k - i\gamma)\sinh(\lambda_j - \lambda_k - i\gamma)} \tag{5.154}$$

have been used.

5.2.4 The scaling limit

The determinants

We want to reduce the expressions above to a standard Fredholm determinant form $\det(1+K) = e^{\sum_n (-1)^{n-1}\,\mathrm{Tr}[K^n]/n}$. For an analytic function (at least on the real axis), one can write

$$\sum_j \frac{f(\lambda_j)}{\prod_{l\neq j}(\lambda_j - \lambda_l)} = \oint \frac{dw}{2\pi i} \frac{f(w)}{\prod_l (w - \lambda_l)} \tag{5.155}$$

This is applied to the matrices (5.141), by considering the variables λ_a, λ_b as two complex variables w, v integrated on a closed contour. In order to do so, one employs the function:

$$A(w) = \prod_l \frac{\sinh\frac{\gamma}{\pi}(w - \mu_l)}{\sinh\frac{\gamma}{\pi}(w - \mu_l - i\pi)} \frac{\sinh\frac{\gamma}{\pi}(w - \lambda_l - i\pi)}{\sinh\frac{\gamma}{\pi}(w - \lambda_l)} \tag{5.156}$$

The latter expression contains both the poles in the values of the λ roots to be summed over and the zeros in the values of the μ roots to be avoided, a fact that allows us to keep the contour of integration under control for every couple of states. Be the reader aware that we are using rescaled variables.

If we consider first the state $\{\mu\}$ to be associated with a (twisted) excited state and the state $\{\lambda\}$ with the ground state, it is sufficient to consider a contour that encircles the real axis. We underline that all the zeros of the factor

$$1 + B^{\omega}_{\mu}(\lambda_a) \to 1 + e^{iZ^{\omega}_{\mu}(w)}$$

are all the real roots and holes in the state $\{\mu\}$. This means that the product

$$\prod_l \sinh\frac{\gamma}{\pi}(w - \mu_l)$$

will cancel all the poles corresponding to real roots, but not the ones corresponding to holes, which will be treated separately.

With w a generic complex variable, having $0 < |\Im\, w| < \pi \min(1, p)$ strictly, it is possible to exponentiate the product and apply the formula (5.91) to the sum of logarithms. It is however necessary to choose the contour in a way to avoid the branch cuts, which is simply done, also numerically, if γ is not close to π or to zero. The result of this is the expression (5.131).

It is necessary to subtract from the sum over poles the unwanted ones corresponding to holes, which is done by using a term like:

$$\frac{\gamma}{\pi} \sum_{holes} \frac{1}{\sinh\frac{\gamma}{\pi}(w - \mu_h)} \frac{A(\mu_h)}{2\pi\, Z'(\mu_h)} K(\mu_h - v) \tag{5.157}$$

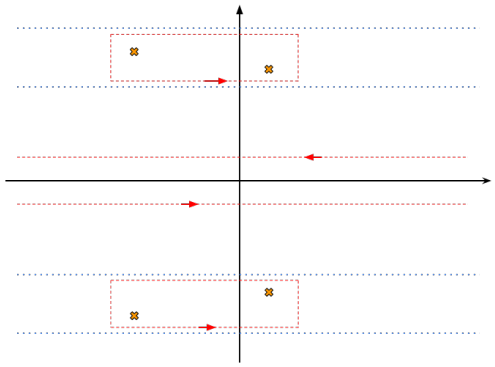


Figure 5.1: contours surrounding roots of the Fermi sea on the real axis and complex roots for $p < 1$. The dotted lines are at $\Im\theta = \pm\frac{\pi}{2}, \pm\pi p$.

as long as the number of holes is of order of unity.

It is also possible to extend our analysis to the case of $\{\mu\}$ being the ground state and $\{\lambda\}$ an excited state: holes need not to be subtracted anymore (the ground state has all the Dirac sea filled), but complex roots outside the contour need to be explicitly added when computing the Fredholm determinant.

This cannot always be done by deforming contours, because of the poles in the kernel at $u - v = \pm\pi, \pm\pi p$. It then follows that roots lying beyond $\min(\frac{\pi}{2}, \frac{\pi p}{2})$ must be treated separately and enclosed in different contours.

Taking into consideration neutral states with rapidities quantized with half-integers [127, 128, 129] the close pairs of complex roots approach their infinite-volume position (5.99) keeping their distance larger than π. Moreover, as was observed in Section 5.1.3, the wide roots do not correspond to any creation operator in the attractive regime, since their presence does not modify the total spin. Then, in the attractive regime, the prescription for the external contours is to surround the region whose imaginary part is $\frac{\pi}{2} < |\Im\theta| < \pi p$. On the other hand, it is known that the antisymmetric soliton-antisoliton states are described by a pair of close roots, so once again, the prescription applies.

The Fredholm determinants are computed on a contour:

$$\det\left[\mathbf{1} - \frac{\gamma}{\pi}\hat{O}\right] = \exp\left\{\sum_n \frac{1}{n}\left(\frac{\gamma}{\pi}\right)^n \oint \frac{dv_1}{2\pi i} \dots \oint \frac{dv_n}{2\pi i} \hat{O}(v_1, v_2)\hat{O}(v_2, v_3)\dots\hat{O}(v_n, v_1)\right\} \tag{5.158}$$

The contours have to surround the real axis; moreover, for excited states, they also need to encircle the complex roots. In principle, one can surround each root by a small circle, taking care to avoid that two points in contours may be separated by $\frac{\pi p}{2}, i\frac{\pi p}{2}$. This is the prescription for the repulsive regime.

In the attractive regime, the contours encircle only close roots, with $|\Im\theta| < \pi\, p$. In particular, it is known that the pair of close roots describing the polarization of a soliton-antisoliton pair, has an imaginary part which reaches the values $\pm i\frac{\pi}{2}$ from above. The same arguments extend to all the close roots quantized with half-integers. Hence, our contours surround the region of the complex plane $\frac{\pi}{2} < |\Im\theta| < \pi p$.

Indicating by a subscript the states that enter in the matrix element, the overlap kernels,

as function of the field theory (rescaled) rapidities, are:

$$\begin{aligned} W_{\lambda,\mu}(x,y) &= \frac{1}{2\pi i}\frac{A(x)}{1+e^{iZ_0(x)}}K_{-\omega}(x-y) \\ W_{\mu,\lambda}(x,y) &= \frac{1}{2\pi i}\frac{A(x)^{-1}}{1+e^{iZ(x)}}K_{\omega}(x-y) - \frac{\gamma}{\pi}\sum_h \frac{1}{\sinh(\frac{\gamma}{\pi}(x-\lambda_h))}\frac{A(\lambda_h)^{-1}}{2\pi Z'(\lambda_h)}K_{\omega}(\lambda_h - y) \end{aligned} \tag{5.159}$$

and the norm kernels

$$\begin{aligned} W_{\mu}(x,y) &= \frac{1}{2\pi i}\frac{1}{1+e^{iZ_0(x)}}K(x-y) \\ W_{\lambda}(x,y) &= \frac{1}{2\pi i}\frac{1}{1+e^{iZ(x)}}K(x-y) - \frac{\gamma}{\pi}\sum_h \frac{1}{\sinh(\frac{\gamma}{\pi}(x-\lambda_h))}\frac{1}{2\pi Z'(\lambda_h)}K(\lambda_h - y) \end{aligned} \tag{5.160}$$

The previous expressions are valid in the scaling limit, provided w is not on the real axis.

More on determinants

Note that the integral operators above are not of trace class. To proceed with the numerical evaluation, one is forced to introduce a cutoff. However, the function A above, which is present in the overlap integrals, tends exponentially to unit value when its argument has large real part, for any couple of the states. It follows that the asymptotic behaviour for large rapidities of the "overlap" kernels is the same. Then, by considering ratios of overlaps as in (5.110), one can argue that the result is independent from the cutoff and the difference of series (5.158) is finite.

Given an integral operator Q and the kernel K, we have that:

$$\det\left[1-(Q-K)\right] = \det\left[1-Q*\frac{1}{1+K}\right]\det\left[1+K\right] \tag{5.161}$$

This can be seen by taking the logarithm of the above expression, expanding and reordering terms:

$$\begin{aligned} \mathrm{Tr}\left[\log\left[1-(Q-K)\right]\right] &= \mathrm{Tr}\Big[K-\frac{1}{2}K^{*2}+\frac{1}{3}K^{*3}-\frac{1}{4}K^{*4}+\frac{1}{5}K^{*5}-\dots \\ &\quad -\left(Q-Q*K+Q*K^{*2}-Q*K^{*3}+Q*K^{*4}\dots\right) \\ &\quad -\frac{1}{2}\left(Q^{*2}-2Q^{*2}*K+2Q^{*2}*K^{*2}+(Q*K)^{*2}-2(Q*K)^{*2}*K\dots\right) \\ &\quad -\frac{1}{3}\left(Q^{*3}-3Q^{*3}*K+3Q^{*3}*K^{*2}+3Q*K*Q^{*2}*K+\dots\right) \\ &\quad -\frac{1}{4}\left(Q^{*4}-4Q^{*4}*K+\dots\right)\Big] \\ &\quad -\dots \\ &= \mathrm{Tr}\left[\log\left(1-Q*\left(1-K+K^{*2}-K^{*3}+\dots\right)\right)\right]+\mathrm{Tr}\left[\log\left(1+K\right)\right] \end{aligned} \tag{5.162}$$

which is used both for the norm and for the overlap kernel, with different Qs and either K_ω or K_0. note that when taking the ratio of amplitudes in the expansion (5.111) the last trace will not play any role, being independent from the states considered. It only enters the computation of $\mathcal{A}_0$.

Considering the kernel in the expression of the norm, we note that the integral on the contour can be written as:

$$\oint \frac{dw}{2\pi i}\frac{K(w-v)}{1+e^{iZ(w)}} = \int_{-\infty}^{\infty}\frac{dw}{2\pi i}\left(\frac{K(w^+-v)}{1+e^{-iZ(w^+)}}+\frac{K(w^--v)}{1+e^{iZ(w^-)}}-K(w^+-v))\right) \tag{5.163}$$

By writing down explicitly the first terms of the Fredholm determinant expansion with this form for the kernel and noting that the function K has no poles on the real axis, it is possible to see that:

$$\begin{aligned}\sum_n \frac{1}{n}\oint\frac{dw_1}{2\pi i}\cdots\oint\frac{dw_n}{2\pi i}\frac{K(w_1-w_2)}{1+e^{iZ(w_1)}}\cdots\frac{K(w_n-w_1)}{1+e^{iZ(w_n)}} &= \mathrm{Tr}\log\left[1-\frac{i\hat{K}}{2\pi}\right] \\ +\sum_n\frac{1}{n}\sum_{\sigma_1\ldots\sigma_n=\pm}\int_{-\infty}^{\infty}\frac{dw_1}{2\pi}\cdots\int_{-\infty}^{\infty}\frac{dw_n}{2\pi}&\left(\frac{G(w_1^{\sigma_1}-w_2^{\sigma_2})}{1+e^{-iZ(w^{\sigma_1})}}\cdots\frac{G(w_n^{\sigma_n}-w_2^{\sigma_1})}{1+e^{-iZ(w^{\sigma_1})}}\right)\end{aligned} \tag{5.164}$$

where we have used the fact that the kernel in the NLIE (5.92) can be expressed [124] in terms of the spin chain kernel as:

$$G(w)=\sum_{n=1}^{\infty}(-1)^{n-1}(-iK)^{\star n}(w)=\int_{-\infty}^{\infty}dqe^{iwq}\frac{-i\hat{K}(q)}{1-i\hat{K}(q)} \tag{5.165}$$

in which the superscript $\star n$ denotes n-times convolution. In order to deal with the overlap determinant, one can similarly set:

$$G_\omega(w)=\sum_{n=1}^{\infty}(-1)^{n-1}(-iK_\omega)^{\star n}(w)=\int_{-\infty}^{\infty}dqe^{iwq}\frac{-i\hat{K}_\omega(q)}{1-i\hat{K}_\omega(q)} \tag{5.166}$$

The Fourier transform is performed by separating the asymptotic behaviour of the function and by regularizing through a damping exponential. In the end of the computation, the regularization parameter is sent to zero, therefore obtaining the "dressed" kernel (5.133). A reasoning analogous to that for the norm leads to the following overlap determinant:

$$\begin{aligned}&\det\left[1-\sum_{\sigma,\sigma'=\pm}U^\lambda_{\sigma\sigma'}(w,v)\right] \\ U^\lambda_{\sigma\sigma'}(w,v) &= \frac{1}{2\pi}\left(\frac{A(w^\sigma)}{1+e^{-\sigma iZ_0(w^\sigma)}}-\delta_{\sigma,+}\left(A\left(w^+\right)-1\right)\right)G_{-\omega}(w^\sigma-v)\end{aligned} \tag{5.167}$$

which is manifestly of trace class due to the presence of the soliton and antisoliton filling fractions and to the asymptotic behaviour of the function A.

The contour indexes will be interpreted as species indexes. This is so because, by deforming the contour up to $|\Im w=\pi/2|$ and following [145], one can define the (complex) soliton and antisoliton pseudoenergies (5.127) as:

$$\begin{aligned}\varepsilon_+(\theta) &= -iZ(\theta+i\frac{\pi}{2}) \\ \varepsilon_-(\theta) &= iZ(\theta-i\frac{\pi}{2})\end{aligned} \tag{5.168}$$

and understand the factor containing the exponentiated counting function as a filling fraction.

$$
\begin{aligned}
W^{\sigma,\sigma'}_{\lambda,\mu}(w,v) &= \frac{1}{2\pi}\left(\frac{A(w+i\sigma\pi/2)}{1+e^{-i\sigma Z_0(w+i\sigma\pi/2)}} - \delta_{\sigma+}\left(A_+(w)-1\right)\right) G_{-\omega}\left(w-v+i\frac{\sigma-\sigma'}{2}\pi^-\right) \\
W_{\lambda,\mu}(\lambda_c,v)^{\sigma'} &= \frac{ResA(\lambda_c)}{1+e^{iZ_0(\lambda_c)}} G_{-\omega}\left(\lambda_c - v - i\frac{\sigma'\pi}{2}\right) \\
W_{\lambda,\mu}(w,\lambda_c)^{\sigma} &= \frac{1}{2\pi}\left(\frac{A(w+i\sigma\pi/2)}{1+e^{-i\sigma Z_0(w+i\sigma\pi/2)}} - \delta_{\sigma+}\left(A_+(w)-1\right)\right) G_{-\omega}\left(w-\lambda_c+i\frac{\sigma}{2}\pi^-\right) \\
W_{\lambda,\mu}(\lambda_c,\lambda_{c'}) &= \left(\frac{ResA(\lambda_c)}{1+e^{iZ_0(\lambda_c)}} - 1\right) G_{-\omega}\left(\lambda_c-\lambda_{c'}\right)
\end{aligned}
\tag{5.169}
$$

$$
\begin{aligned}
W^{\sigma,\sigma'}_{\mu,\lambda}(w,v) &= \frac{1}{2\pi}\left(\frac{A^{-1}(w+i\sigma\pi/2)}{1+e^{-i\sigma Z(w+i\sigma\pi/2)}} - \delta_{\sigma+}\left(A_+^{-1}(w)-1\right)\right) G_{\omega}\left(w-v+i\frac{\sigma-\sigma'}{2}\pi^-\right) \\
W^{\sigma'}_{\mu,\lambda}(\lambda_h,v) &= -\frac{A^{-1}(\lambda_h)}{Z'(\lambda_h)} G_{\omega}\left(\lambda_h - v - i\frac{\sigma'\pi}{2}\right) \\
W^{\sigma}_{\mu,\lambda}(w,\lambda_h) &= \frac{1}{2\pi}\left(\frac{A^{-1}(w+i\sigma\pi/2)}{1+e^{-i\sigma Z(w+i\sigma\pi/2)}} - \delta_{\sigma+}\left(A_+^{-1}(w)-1\right)\right) G_{\omega}\left(w-v+i\frac{\sigma-\sigma'}{2}\pi^-\right) \\
W_{\mu,\lambda}(\lambda_h,\lambda_{h'}) &= -\left(\frac{A^{-1}(\lambda_h)}{Z'(\lambda_h)} - 1\right) G_{\omega}\left(\lambda_h-\lambda_{h'}\right)
\end{aligned}
\tag{5.170}
$$

It is then possible to apply again the partial summation of some families of terms in the Fredholm determinant series along the route described above. Unfortunately, it is not as easy as in the previous case to Fourier transform and sum over all the terms, so that the result is given in the rather implicit form (5.134).

The expressions above imply explicit subtraction of holes in the ket and summation of the complex roots of the bra.

The prefactor

We start by the analysis of the factor (5.154). Here the complication lies in the double product, but there are no conceptual difficulties in exponentiating this expression and performing the sum of the resulting logarithms with the procedure described above, since the arguments of the logarithms never cross the cut. The first step is

$$
\begin{aligned}
&\sum_{j,k}\left(\log\frac{\sinh(\lambda_j-\mu_k-i\gamma)}{\sinh(\lambda_j-\lambda_k-i\gamma)} + \log\frac{\sinh(\mu_j-\lambda_k-i\gamma)}{\sinh(\mu_j-\mu_k-i\gamma)}\right) \\
&= \sum_j\Big\{ -\sum_{\sigma=\pm}\frac{\sigma}{2\pi i}\int dx\left(\Delta\star\log'\frac{1+e^{i\sigma Z_0}}{1+e^{i\sigma Z}}\right)\log\frac{\sinh(\lambda_j-x^\sigma-i\gamma)}{\sinh(\mu_j-x^\sigma-i\gamma)} \\
&+ \int du\hat{\Delta}(u)\log\frac{\sinh(\lambda_j-u-i\gamma)}{\sinh(\mu_j-u-i\gamma)}
\end{aligned}
\tag{5.171}
$$

Then one is to apply again (5.91) to the j index. After passing to the scaling limit (5.69) we find the result (5.124).

The complex roots are, by definition, away from the real axis. Since in the product defining the complex root factor (5.152) there appear all and only the real solutions, it is natural to consider the logarithm and chose a suitable contour around the real axis to perform the sum over roots. The resulting expression is given in (5.122).

We turn to the analysis of the factor containing the position of the holes. The counting function itself may be non monotonic in some region of the real axis for some class of states and at small volumes $mL \sim O(1)$, a circumstance which is connected with the appearance of *special* roots when the counting function also crosses a quantization point within that region. We assume that this is not the case, even if, in principle, such a case can be worked around by splitting the sum over different regions in which the counting function is monotonically increasing. Having written the product (5.150) in terms of positive functions, we can now take the logarithm and perform the scaling limit.

Such limit is somewhat simplified by the fact that the sum is performed only over real solutions of the (5.60). It can be performed by standard techniques, but some details are in order. After applying the summation procedure, one of the terms has the form

$$\mathcal{I}_0 = \int \frac{du}{2\pi} \left(\frac{N}{\cosh(u-\Theta)} + \frac{N}{\cosh(u+\Theta)} \right) \log \frac{\varphi_0(x,u)}{\varphi(x,u)} \tag{5.172}$$

This integral is similar to the ones needed for the computation of energy and momentum ([135, 127, 128, 129], see also [136, 144]): one uses the fact that the counting function has a well defined limit when the argument is sent to infinity

$$\begin{cases} Z(+\infty) &= N\pi + \frac{p-1}{p+1}\pi S + 2\pi \operatorname{sgn}(p-1) M_{W\downarrow} + 2\omega \\ Z(-\infty) &= -N\pi - \frac{p-1}{p+1}\pi S - 2\pi \operatorname{sgn}(p-1) M_{W\uparrow} + 2\omega \end{cases} \tag{5.173}$$

where $M_{W\downarrow}$ and $M_{W\uparrow}$ are for the number of wide roots below and above the real axis. It follows that:

$$\mathcal{I}_0 = \operatorname{sgn}(p-1) M_W \tag{5.174}$$

This integral enters both in the evaluation of the "hole" factor and in the product (5.151). Here below, we report the result of the summation an the first index:

$$\begin{aligned} \sum_{j,k} \log \frac{\varphi(\tilde\lambda_l, \tilde\lambda_k)\varphi_0(\mu_j,\mu_k)}{\varphi(\tilde\lambda_l,\mu_k)\varphi_0(\tilde\lambda_j,\mu_k)} &= \sum_j \Big\{ \sum_{\sigma=\pm} \int \frac{du\,dx}{2\pi i\sigma} \log' \frac{1+e^{i\sigma Z(u^\sigma)}}{1+e^{i\sigma Z_0(u^\sigma)}} \Delta(u-x) \log \frac{\varphi_0(\tilde\mu_j, x^\sigma)}{\varphi(\tilde\lambda_j, x^\sigma)} \\ &\quad + \sum_\alpha c_\alpha \int du G(\lambda_\alpha - u) \Big(\log\varphi(\tilde\lambda_j, u^-) - \log\varphi_0(\tilde\mu, u^-) \Big) \Big\} \end{aligned} \tag{5.175}$$

the second step, again by taking into account the asymptotic behaviour (5.173), can be analogously performed and yields (5.123) as a result.

5.2.5 Conclusions

We have presented an exact expression for the generating function of connected correlation functions on a cylinder, with compactified space direction and infinite time direction, for the sine-Gordon quantum field theory.

To take into account corrections which enter as exponentials in the size, the knowledge of the spectrum and of the form factors of the theory in infinite volume is not enough. To circumvent this problem, the computations were carried on in the framework of the Destri-De Vega lattice regularization, formulated in terms of an inhomogeneous XXZ spin chain.

In this framework, the problem was similar to the study of the expectation value of the magnetization of the spin chain in a given interval: due to the available results for this model,

the most important of which is the solution of the quantum inverse scattering, we have written the matrix elements and performed the appropriate scaling limit to obtain the result (5.118).

The expression is written as an expansion on the states of the field theory in finite volume of the generating functions of form factors. The determination of such states relies on the ability of solving self-consistently the Destri-De Vega nonlinear integral equation for all the allowed source terms, which is in general a difficult task and limits for the moment the practical applicability of the method to some classes of finite volume states.

Having shown its relevance in the computation of correlation functions, we hope to be able to extend the analysis of the nonlinear integral equation in the future. Further work would be moreover required to explore the advantages and limitations of this kind of formalism in the actual computation of correlation functions, as well as to identify the features which may be generally valid for other integrable field theories.

Summary

In this work, we have studied three problems of current interest in condensed matter and statistical physics. Despite looking very different at first sight, they are intimately connected from the point of view of their algebraic structure, as shown in Chapter 1.

The Josephson currents between two Richardson Hamiltonians coupled by a tunnelling term have been analysed in Chapter 2. By the use of the exact solution of the separate systems we were able to implement an algorithm that allows to study the exact dynamics of the system after the tunnelling term is switched on. In particular, the phase relation among the two parts of the system and the population-phase diagram were studied. The question of scaling with the size and the effect of the charging energy are for the moment open, yet the approach is promising and allows useful insight on the physics of Josephson currents. This formalism serves as a basis for the study of finite–size effects in BCS superconductors.

An example of exactly solvable interacting and disordered system was presented in Chapter 3. The properties of its eigenstates were studied with respect to the interplay between disorder and interaction and to the many-body localization characteristics. It was shown how a delocalization transition happens only at the trivial noninteracting point and how a delocalized, yet nonergodic phase is set. It is an interesting question, not fully addressed yet, to study how the breaking of integrability affects this scenario.

The sine-Gordon integrable field theory was studied in Chapter 5. An exact formula for the generating function of connected correlation functions in arbitrary finite volume was derived, by the use of an integrable lattice regularization of the theory. This expression encodes the exponential contributions that the asymptotic approaches presented so far were not able to include. Nevertheless, its use appears to be rather cumbersome for numeric, at the moment, and will find its application in the study of single form factors.

All these achievements were possible by the use of the algebraic Bethe ansatz technique. In particular, the recent progresses in the computation of the exact matrix elements have been widely exploited in this thesis. The common structure shared by both integrable lattice models and integrable field theories is ultimately at the roots of part of the work, and is worth further attention. Such a beautiful algebraic structure proved to be versatile enough to tackle very different problems, arising in sectors of physics which are far one from the other.

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Grazie

... and I can't forget, I can't forget
I can't forget, but I don't remember who
(Leonard Cohen, I can't forget)

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