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Some Aspects of Anyon Thermodynamics and Chern-Simons Theory

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To my mother Maria Grazia

who prematurely left us,

and to the little, always smiling, Angelica

Foreword

My years at SISSA have passed too quickly in my temporal frame, and new experiences have left a firm mark on my formation. I have shared this adventure with many people, which I cannot list all here only for conciseness reasons. My thanks to my two supervisors, Prof. Giuseppe Mussardo and Dr. Andrea Trombettoni, will hardly be enough for having gradually taught me, among all other things, how to draw up a scientific paper in an interesting way, how to communicate within the actual scientific community, and to present my results in front of a professional audience. We had several discussions, and they were always prompt to advise me without ever imposing their point of view, or obstructing me in taking my decisions. Thanks also for their letters of support on behalf of me. I am extremely grateful to Prof. Enore Guadagnini working at the University of Pisa: his experience and his patience with me have been irreplaceable in several critical moments. I am furthermore deeply grateful to a series of colleagues whom I met here or in various occasions, and with whom I discussed in a proficient way. Thanks to all my friends and colleagues in Trieste, and to my geometrically closest workmates Pietro Giavedoni, Francesco Buccheri and Xiaoquan Yu: I owe my achievements to the warm atmosphere and to the support that all of them selflessly offered to me. Thanks to my sister, my parents, and all close members of my family, for their constant encouragement and esteem. I feel indebted also with Prof. Luigi Martina (Universita' del Salento, Arnesano di Lecce) and with Dr. Alessandro Michelangeli (LMU, Munich) for their advices in some decisive moments, and for having often embodied the reassuring shadow of an older colleague behind me. Special thanks to Marilena, with whom I share my whole life since last year, for all her love, support, hints, patience, ideas, and for her teaching in the broadest sense.
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Contents

Abstract

In Chapter 1, after having introduced the two-dimensional world and its exotic braiding statistical features, we discuss the arising of anyonic statistics, Wilczek's flux-tube model and the quantum mechanical features of an ideal anyon system. In Chapter 2 the Abelian and non-Abelian quantum Chern-Simons theory is presented through its deep relation with anyonic statistics, and the Verlinde's model of non-Abelian Chern-Simons particles and its non-Abelian braiding statistics are widely discussed. Chapter 3 starts with a mathematical foreword aimed to rub up the concept of homology groups, protagonist of the results at the heart of the following pages. An introduction to the field theory approach to the Abelian Chern-Simons theory follows, after which several relations between the homological features of closed worldlines of pure gauge-Abelian CS particles and their Wilson line expectations (with respect to a pure CS field-theoretical action) are investigated in detail. We consider the link invariants defined by the quantum Chern-Simons field theory with compact gauge group $U(1)$ in a closed oriented 3-manifold M. The relation of the abelian link invariants with the homology group of the complement of the links is discussed. We prove that, when M is a homology sphere or when a link -in a generic manifold M- is homologically trivial, the associated observables coincide with the observables of the sphere S^3. We show that the $U(1)$ Reshetikhin-Turaev surgery invariant of the manifold M is not a function of the homology group only, nor a function of the homotopy type of M alone. In Chapter 4 we study the thermodynamical properties of an ideal gas of non-Abelian Chern-Simons particles of the Verlinde's model, and we compute the second virial coefficient, considering the effect of general soft-core boundary conditions for the two-body wavefunction at zero distance. The behaviour of the second virial coefficient is studied as a function of the Chern-Simons coupling, the isospin quantum number and the hard-core parameters. Expressions for the main thermodynamical quantities at the lowest order of the virial expansion are also obtained: we find that at this order the relation between the internal energy and the pressure is the same found (exactly) for 2D Bose and Fermi

ideal gases. A discussion of the comparison of obtained findings with available results in literature for systems of hard-core non-Abelian Chern-Simons particles is also supplied. In Chapter 5 we determine and study the statistical interparticle potential of the same system of NACS particles, comparing our results with the corresponding results of an ideal gas of Abelian anyons. In the Abelian case, the statistical potential depends on the statistical parameter and it has a "quasi-bosonic" behaviour for statistical parameter in the range $(0, 1/2)$ (non-monotonic with a minimum) and a "quasi-fermionic" behaviour for statistical parameter in the range $(1/2, 1)$ (monotonically decreasing without a minimum). In the non-Abelian case the behavior of the statistical potential depends on the Chern- Simons coupling and the isospin quantum number: as a function of these two parameters, a phase diagram with quasi-bosonic, quasi-fermionic and bosonic-like regions is obtained and investigated. Finally, using the obtained expression for the statistical potential, we compute the second virial coefficient of the NACS gas, which correctly reproduces the results available in literature. Chapters 3,4,5 present original results appeared in [1, 2, 3]. After that, our findings are discussed in the Conclusions, and some computations mentioned in the main text find place in the Appendices.

Basics of Two-Dimensional Statistics

This introduction is devoted to explain the arising of fractional braiding statistics in two space dimensions, whereas three- and higher-dimensional systems can harbour only ordinary Bose and Fermi statistics [4, 5, 6, 7, 8, 9, 10]. We will first consider the fractionalization of the spin eigenvalues in units of $\hbar$, and then the fractionalization of the statistics. The reason for the latter one lies in the fact that coincident points for two or more particles in two dimensions are singular points in the space of their configurations, so that they must be excluded when considering its subset relevant to braiding statistics. Hence the braid group will take the place of the ordinary permutation group in all the issues in exam. We will present a dynamic model for anyons, in which they behave as point particles pinned with a one-dimensional flux-tube threading the 2D plane hosting the particles: anyons correspond in such a way to point charged vortices. So let us start with a motivation for the fractional spin. In (d+1) dimensions, $d \geq 3$, the angular momentum algebra is a non-commutative one; by setting $\hbar = 1$:

$$[S_i, S_j] = i\, \epsilon_{ijk}\, S_k, \tag{1}$$

ϵ_{ijk} being the completely antisymmetric tensor, while in (2+1) dimensions the angular momentum algebra is commutative, being generated by the only available generator S_3 for the 1-parameter group of the rotations in the plane. The absence of non-trivial commutation relations among the generators of the rotations explains the absence of any quantization of the angular momentum. Let us just observe that in (1+1) dimensions there is not any rotation axis, so that there is not even a notion of spin in the one-dimensional case. From relativistic quantum field theory we know that statistics of the particles is constrained by their spin [11, 12, 13], i.e. particles with half-integer spin obey Fermi-Dirac statistics, particles with integer spin obey Bose-Einstein statistics. This leads to the insight that in (2+1) dimensions the particles may instead exhibit fractional statistics. Now we will see that this is indeed the case, by making use of topological considerations. At this point a conceptual remark is in order. The term quantum statistics has to do with the phase factor acquired by the quantum wavefunction when two identical, indistinguishable particles are adiabatically transported giving rise, at the end of the process, to an exchange of their positions. We are ready to explore the arising of fractional statistics. We consider the configuration space of a system of N identical particles, being denoted by X the single-particle configuration space. The indistinguishability of all the particles implies the coincidence between a configuration $\vec{x} = (\vec{x_1}, \vec{x_2}, \cdots, \vec{x_N}) \in X^N$ and the generic configuration $\vec{x'} \in X^N$ obtained by any permutation $p \in S_N$ applied to $\vec{x}$, because $\vec{x}$ and $\vec{x'}$ describe the same physical configuration. Hence the true configuration space is represented by the quotient space X^N/S_N, which is a space locally isomorphic to X^N, except at its singular points. Points in X^N/S_N which correspond to a coincidence of the positions of some particles, called *coincident points*, are singular points of X^N/S_N. This is one of its differences with respect to the space X^N; the coincident points must be excluded in order to discuss the adiabatical exchange of particles and consequently their statistics. This crucial point, maybe apparently elementary in hindsight, was neglected during all the development of quantum mechanics until the appearance of the seminal work by Leinaas and Myrheim [14]. This consideration enables us to realize the deep difference between the configuration space in two dimensions and in higher dimensionality. For instance an analysis of the simple two-particle case shows how the removal of the coincident points makes the spaces multiply connected (in a denumerable way) in the former case, while in the latter it is only double connected. As a consequence, only in two dimensions we can physically distinguish any arbitrary integer number of windings of particles around each other (let us say, of a particle around the origin in a two-particle center of mass system), counting the number with its orientation. Formally this distiction is represented by a different first homotopy group π_1, defined as the group of "inequivalent" paths (in a topological sense) passing through a given point in configuration space, with the obvious definition of group multiplication and inverse [15, 16, 17]. If the one-particle available coordinate space is the Euclidean $d-$dimensional space R^d, we write for instance the first homotopy groups for $N = 2$ particles, corresponding to the two cases:

$$\pi_1^{d=2} = \pi_1 \left(\frac{(R^2 - \{0\})}{Z_2} \right) = \pi_1(RP_1) = Z; \quad (2)$$

$$\pi_1^{d\geq 3} = \pi_1 \left(\frac{(R^d - \{0\})}{Z_2} \right) = \pi_1(RP_{d-1}) = Z_2. \quad (3)$$

Similarly, for a system of N identical particles the configuration space is

$$X^N/S_N = \frac{(R^d)^N - \Delta}{S_N}, \tag{4}$$

where Δ is the set of coincident points

$$\Delta \equiv \{(\vec{r}_1, \cdots, \vec{r}_N) : \vec{r}_I = \vec{r}_J \text{ for some } I \neq J\}, \tag{5}$$

and its first homotopy group is [18, 19]

$$\pi_1^{d=2}(X^N/S_N) = \pi_1(RP_1) = B_N; \tag{6}$$

$$\pi_1^{d\geq 3}(X^N/S_N) = S_N, \tag{7}$$

where B_N denotes the braid group of N objects and S_N, the permutation group of N objects, is one of its finite subgroups. As the permutation group is the signature of the Bose and Fermi statistics, which are associated to its only two one-dimensional representations (respectively the trivial and the alternating ones), instead the braid group corresponds to an infinite family of fractional statistics (anyonic statistics), in a one-to-one correspondence with the continuous one-parameter family of possible one-dimensional group representations.

Chapter 1

Aspects of non-relativistic anyons

1.1 Properties of anyons

• The one-dimensional representations of the braid group are labeled by a continuous parameter θ related to the phase factor gained by the global wavefunction when two particles far from the others are exchanged, and these representations are inequivalent up to a periodicity modulo 2π of this parameter. The phase due to the braiding (= adiabatic exchange) of two particles depends in principle on the position of all the other particles. Indeed if an anyon is wrapped in a closed orbit around another, and its orbit includes by chance another anyon (identical to the previous ones), then the resulting phase factor is $e^{2i\theta}$ instead of being $e^{i\theta}$.

• Anyons violate the symmetries of parity (P) (defined in two space dimensions by the transformation $x \to -x$ and $y \to y$) and the time reversal invariance (T) if $0 < \alpha < 1$, where statistical parameter $\alpha \equiv \theta/\pi$, $e^{i\theta}$ being the multiplicative phase factor taken by the wavefunction under adiabatic exchange of two anyons. In fact the effect of spatial reflection or time reversal corresponds to replace clockwise and counter-clockwise windings each other. These opposite windings, as already pointed out, are equivalent in three or higher space dimensions, but inequivalent in the two-dimensional case for homotopical reasons, whence the lack of these symmetry for anyons.

• Anyonic statistics and parastatistics are two completely different generalizations of ordinary statistics. The former has to do with the continuous family of the one-dimensional representations of the braid group, while the latter is a statistics defined by means of the higher dimensional representation of the permutation group. Consequently, anyons exist only in two dimensions, while parastatistics are defined for whatever dimension $d \geq 2$. Nevertheless, one can imagine higher-dimension representations also for the braid group, and the result of this generalization are the so-called non-Abelian anyons. This conceptual framework is enlightened in Tab.1.1

In order to emphasize the genuine quantum nature of this model, we present here in the full detail the arising of fractional statistics, which in the previous discussion was referred to simply as an eligible eventuality. Let us consider a particle of mass m and charge q moving on a circular ring of radius R. Perpendicularly to the plane, the ring is assumed to be threaded by a point-like solenoid carrying flux Φ. The dynamics of the particle will be governed by the Lagrangian function:

$$L = \frac{1}{2} m R^2 \dot{\phi}^2 + \frac{q\Phi}{2\pi} \dot{\phi} \tag{1.1}$$

The associated canonical angular momentum is

$$p_\phi = m R^2 \dot{\phi}^2 + \frac{q\Phi}{2\pi} \dot{\phi} \tag{1.2}$$

and the Hamiltonian

$$H = \frac{1}{2mR^2} \left(p_\phi - \frac{q\Phi}{2\pi} \right)^2 \tag{1.3}$$

Its eigenfunctions are

$$\psi_n = e^{in\phi}, \tag{1.4}$$

and its eigenvalues are

$$E_n = \frac{1}{2mR^2} \left(n - \frac{q\Phi}{2\pi} \right)^2 \tag{1.5}$$

By considering that the energies are proportional to the square of the kinetic angular momentum (with proportionality constant naturally represented by one half times the inverse of the moment of inertia), it is understood that the interaction due to the magnetic flux produces allowed kinetic angular

	(3+1)dim$\rightarrow S_N$	**(2+1)dim**$\rightarrow B_N$
1D repr.	BOSE/FERMI statistics	Abelian Anyons
Higher-dim repr.	PARASTATISTICS	NON-Abelian Anyons

Table 1.1: Classification of homotopically inequivalent paths of N strings in 2D and 3D

momenta spaced by integers (in units of Planck's constant), but subjected to an uniform shift by $q\Phi/2\pi$ from integer values, so that we have built fractionalization of the angular momentum. Let's come to point out the crucial difference from the corrisponding classical system. All the Lagrangians (1.1) would produce the same *classical* equations of motion at varying the external parameter Φ, because the term by which they differ is a total derivative, which cannot affect the classical variational principle. Instead we have seen that they lead to different quantum theories, which can be understood from the point of view of canonical quantization. The canonical quantization of the theory also requires to impose commutation relations between the coordinates and their conjugate momenta. So the content of the commutation relations is changed if the definition of these momenta is changed, whether or not the equations of motion are affected. In this quantization problem we have therefore seen how just such a modification can affect the physical consequences of a given classical Lagrangian.

1.1.1 Path-integral point of view

We have mentioned so far the point of view of canonical quantization. But this system appears even more tantalizing from the point of view of path integral quantization: in the latter quantization prescription, transition amplitudes are computed by adding the contributions of all possible paths, by weighing each of them according to the exponential of the classical action along the path; thus one would have the impression that the transition amplitudes should be determined by the classical action only. We will see that the geometric phase plays a crucial role in this problem. Our point is that the classically ignorable $\dot{\phi}$ term is not ignorable in the quantum theory. Let us discover the effect of this term, path by path. We limit ourselves to compare paths that begin at a common position ϕ_1 at time t_1 and end at the same position ϕ_2 at time t_2, since these are the kinds of couples of terms which can interfere. The effect of the $\dot{\phi}$ term is to weight their relative contribution by

$$\exp\left[i\frac{q\Phi}{2\pi}\left(\int_{\text{path 1}}\dot{\phi}\,dt - \int_{\text{path 2}}\dot{\phi}\,dt\right)\right] \equiv \exp\left[i\frac{q\Phi}{2\pi}\delta\phi\right] \tag{1.6}$$

Since "path 1" and "path 2" must share their respective endpoints, $\delta\phi$ must be a mutiple of 2π, by a factor given by the difference between the number of windings of the first and the second path around the center. In different terms, $\delta\phi/2\pi$ is the number of windings around the solenoid for the closed path consisting in "path 1" from t_1 to t_2 followed by the inverse of the second path back from t_2 to t_1. The possible closed paths of this kind belong to distinct, disconnected classes, labeled by (integer) winding numbers. The classical Lagrangian does not suggest any way of weighing the relative contributions of disconnected classes of paths. In fact, the classical equations of motion follow from a variational principle that involves only comparison among infinitesimally nearby paths, and cannot account of global differencies like those between disconnected classes of paths. An ambiguity in the path integral quantization procedure follows as a consequence of that. We can say that a single classical Lagrangian can lead to various quantum theories whenever the first homotopy group of the configuration space in non-trivial. The exponential prescription for the weighing of the paths is in accordance with the general rule of quantum mechanics for which the amplitude for the composition of two paths must be the product of the amplitude for each path separately. By equipping the above mentioned set of disconnected classes paths with the product structure represented by the composition of representative paths, we just get the first homopoty group. If we are assigning extra numerical factors α_π to the paths, we must demand that they obey the rule

$$\alpha_{\pi_1 \circ \pi_2} = \alpha_{\pi_1} \cdot \alpha_{\pi_2}, \tag{1.7}$$

so these factors form a (one-dimensional) representation of the first homotopy group. They can be considered geometric phases. The geometric phase between nearby position eigenstates $|\phi\rangle$ and $|\phi + \Delta\phi\rangle$ results to be

$$\exp\left[i\frac{q\Phi}{2\pi}\Delta\phi\right] \tag{1.8}$$

This phase is locally but not globally integrable, because after an adiabatic round of 2π there is an accumulation of a phase $e^{iq\Phi}$. So the geometric phase around an homotopically non-trivial path in configuration space parametrizes the ambiguity in quantization.

A crucial consequence of that is the impossibility of defining the wave function on ordinary configuration space: indeed the position eigenstates for ϕ and $\phi + 2\pi$ differ by a phase factor, so we should define the wave function with ϕ running from $-\infty$ to ∞, with the boundary condition

$$\psi(\phi + 2\pi) = e^{iq\Phi}\psi(\phi) \tag{1.9}$$

In general the wave function will live (if there are not internal degrees of freedom) on the universal covering space of configuration space, and will obey boundary conditions relating points that project to the same point in configuration space.

1.1.2 Flux-Tube Model

The fractional statistics of anyons can be represented through the dynamical model called flux tube model, first proposed by Frank Wilczek in 1982 [20]. We start by considering the Lagrangian for two identical massive bosons subjected to a statistical interaction term

$$L = \frac{m}{2}(\dot{\mathbf{r}}_1^2 + \dot{\mathbf{r}}_2^2) + \alpha \frac{d\phi}{dt}, \tag{1.10}$$

where $\alpha = \theta/\pi = q\Phi/2\pi$. Here α is the statistical parameter and ϕ is the relative angle between the identical particles. The total Lagrangian can either be interpreted as the Lagrangian for two interacting bosons or as the Lagrangian for two non-iteracting anyons. This interpretation makes clear why also the simplest problem of non-interacting anyon gas is nontrivial. Indeed a non-interacting anyon gas problem is equivalent to an interacting Bose (or Fermi) gas problem, which is in general quite difficult to solve. We will use this equivalence to study the statistical mechanics of a non-interacting anyon gas and we will see how just a few two-anyon problems have been solved exactly, while for no multi-anyon ($N \geq 2$) problem a complete solution has been produced. We separe the center of mass (written in uppercase) and the relative (in lowercase) motions:

$$\mathbf{R} = (\mathbf{r}_1 + \mathbf{r}_2)/2 \rightarrow \ \mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2 \tag{1.11}$$

$$\mathbf{r} = (\mathbf{r}_1 - \mathbf{r}_2) \rightarrow \ \mathbf{p} = (\mathbf{p}_1 - \mathbf{p}_2)/2 \tag{1.12}$$

so that the Lagrangian takes the form $L = L_R + L_r$, where

$$L_R = m\dot{\mathbf{R}}^2 \tag{1.13}$$

$$L_r = \frac{m}{4}\dot{\mathbf{r}}^2 + \alpha\dot{\phi} = \frac{m}{4}(\dot{r}^2 + r^2\dot{\phi}^2) + \alpha\dot{\phi} \tag{1.14}$$

The center of mass motion is free ($H_R = P_R^2/4m$) and independent of the statistical parameter α. The same would stand also for N anyons. Thus to understand the anyon dynamics, it is enough to concentrate on the Hamiltonian for the relative motion, whose two-body Hamiltonian is

$$H_r = \frac{p_r^2}{m} + \frac{(p_\phi - \alpha)^2}{mr^2} \tag{1.15}$$

The Lagrangian for N non-interacting anyons generalization is

$$L = \frac{m}{2}\sum_{i=1}^{N}\dot{\mathbf{r}}_i^2 + \alpha\sum_{j\neq i}^{N}\frac{d\phi_{ij}}{dt} \tag{1.16}$$

where $\phi_{ij} = \arctan[(y_i - y_j)/(x_i - x_j)]$. In the sequel of this Subsection, the arising of the statistical term in the Lagrangian (1.10) is justified by means of the dynamical realization after which the flux tube model is named. One should imagine to attach *fictitious* "electric" charge and a delta-function "magnetic" flux to each particle. This flux tube corresponds to a *fictitious* vector gauge potential

$$a_i(\vec{r}) = \frac{\Phi}{2\pi}\frac{\epsilon_{ij} r_j}{r^2}. \tag{1.17}$$

By minimal prescription, the Hamiltonian for any one of the particles is

$$H = \frac{1}{2m}(p_i - ea_i)^2 \tag{1.18}$$

A dynamical effect of the point flux tube is the production of the following kind of weight factors for the paths of the path integral prescription

$$\exp\left(\int L_{int} dt\right) = \exp\left(iq\int \vec{v}\cdot\vec{a}dt\right) = \exp\left(iq\int \vec{a}\cdot d\vec{x}\right) \tag{1.19}$$

due to the interaction of the particle with the gauge field. By replacing $\vec{a}$ with its expression (1.17) for our case, we have the weight

$$e^{iq\int \vec{a}\cdot d\vec{x}} = e^{iq\frac{\Phi}{2\pi}\Delta\phi}, \tag{1.20}$$

which is nothing but the factor needed to implement the fractional statistics, presented in (1.8). Hence, we have seen that the dynamical effect of attaching the fictitious charge and delta-function flux to particles is a faithful implementation of fractional statistics, and that $q\Phi/2$ (or $q\Phi/2\hbar c$, if not using natural units) can be identified with the anyon paramete θ (the apparent factor of two appears because each particle moves in the potential of the other). The generalization to the N-particle case is needed for our purposes. By observing that the gauge potential (1.17) can be rewritten as

$$\mathbf{a}(\mathbf{r}) = \frac{\Phi}{2\pi}\nabla\theta \tag{1.21}$$

It follows that the N-particle generalization is

$$\mathbf{a}_i(\mathbf{r}_i) = \frac{\Phi}{2\pi}\sum_{j\neq i}\nabla_i\theta_{ij} = \frac{\Phi}{2\pi}\sum_{j\neq i}\frac{\hat{\mathbf{z}}\times(\mathbf{r}_i - \mathbf{r}_j)}{|\mathbf{r}_i - \mathbf{r}_i|^2}, \tag{1.22}$$

where $\theta_{ij} = \arctan[(y_i - y_j)/(x_i - x_j)]$ is the relative angle between the particles i and j. We notice that the charge in each anyon sees the vector potential due to the flux tubes in all the other anyons. The Hamiltonian for N noninteracting anyons is the sum of the N terms $(\mathbf{p}_i - e\mathbf{a}_i)^2/2m$, $i = 1, \cdots, N$, deriving from the minimal prescription, hence finally

$$H = \frac{1}{2m}\sum_{i=1}^{N}\mathbf{p}_i^2 - \frac{\alpha}{2m}\sum_{j\neq i}^{N}\frac{\mathbf{L}_{ij}}{r_{ij}^2} + \frac{\alpha^2}{2m}\sum_{j,k\neq i}^{N}\frac{\mathbf{r}_{ij}\cdot\mathbf{r}_{ik}}{r_{ij}^2 r_{ik}^2} \tag{1.23}$$

where natural units have been used, $\alpha = \theta/\pi = q\Phi/2\pi$, $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$ and $\mathbf{L}_{ij} = (\mathbf{r}_i - \mathbf{r}_j)\times(\mathbf{p}_i - \mathbf{p}_j)$. So the statistical transmutation can be described by stating that the N-anyons ideal system is equivalent to an interacting boson (or fermion) system with a long range *vector* interaction. This Hamiltonian has (only) two-body and three-body interaction terms. We are not aware of a proof for the existence of the virial expansion for this system, because of its long range interaction; notwithstanding, the virial expansion seems to hold at least for a noninteracting gas. Some interacting two-body problems (beyond the noninteracting itself) are exactly solvable, while no multianyonic ($N \geq 3$) one has been exactly solved as yet, just because of the presence of the three-body interaction term.

Pictorially, an anyon can therefore be seen as a point charged particle with an infinitely long one-dimensional tube piercing the charge; the flux can take any value and the anyon behaves as a point charged vortex. The flux-tube model just described has an only formal identification with anyons, in other words the invoked gauge fields are completely fictitious.

1.1.3 Gauge choice

The Hamiltonian (1.23) can indifferently act on the spaces of symmetric or antisymmetric wave functions. The theories $(B, \alpha + 1)$ and (F, α), where B, F denote the space of symmetric/antisymmetric functions and α is the statistical parameter, are gauge equivalent theories, because of the simple gauge transformation connecting the respective wavefunctions:

$$\psi_F(\alpha) = U\,\psi_B(\alpha + 1) \tag{1.24}$$

where

$$U = \prod_{i>j}\exp\left(i\,\phi_{ij}\right) \tag{1.25}$$

This gauge choice is called the **magnetic** gauge (or *boson* gauge). An alternative possibility is represented by the **anyon** gauge, where the Hamiltonian is that for free particles, i.e.

$$H_A = \sum_{i=1}^{N}\frac{\mathbf{p}_i^2}{2m} \tag{1.26}$$

The anyon gauge entails a simpler presentation for the Hamiltonian, but the wave functions are multi-valued in this gauge (for the truly fractional cases), while those in the magnetic or bosonic gauge are single valued. The relation between these two gauges reads

$$\psi_A(\alpha) = U^{\alpha}\psi(\alpha) \tag{1.27}$$

where the wave function for boson/fermion-based anyons appears in the left-hand side above, respectively for $\psi(\alpha) = \psi_B(\alpha), \psi_F(\alpha)$.

1.2 Anyonic Quantum Mechanics

The simplicity in deriving properties of an ideal gas of bosons/fermions is a result of the possibility of writing down the N-particle wave function as the product of single particle wave functions, with appropriate symmetry or amtisymmetry factor respectively. The situation is different for anyon gases. Even in the two-anyon case, multi-particle wave functions cannot, in general, be decomposed in terms of single particle wave functions. We start right by considering the two-anyon case.

1.2.1 Two noninteracting anyons

We previously saw that the relative Hamiltonian, obtained by factoring out the center of mass term from the full two-body Hamiltonian, is

$$H_r = \frac{p_r^2}{m} + \frac{(p_\phi - \hbar\alpha)^2}{mr^2} \tag{1.28}$$

The corresponding Schrödinger equation reads in cylindrical coordinates:

$$\left[-\frac{\hbar^2}{m}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) + \frac{\hbar^2}{mr^2}\left(i\frac{\partial}{\partial\phi} + \alpha\right)^2\right]\psi(r,\phi) = E\,\psi(r,\phi) \tag{1.29}$$

which is separable in r and ϕ by setting $\psi(r,\phi) = R(r)F_l(\phi)$.The angular equation takes the form

$$\left(i\frac{\partial}{\partial\phi} + \alpha\right)^2 F_l(\phi) = \lambda F_l(\phi) \tag{1.30}$$

Since we are working in the magnetic (= boson) gauge, the angular component has to be periodic modulo π, hence the solution of (1.30) consistent with the bosonic boundary condition is

$$F_l(\phi) = e^{il\phi}, \quad l = 0, \pm 2, \pm 4, \cdots, \quad \lambda = (l-\alpha)^2 \tag{1.31}$$

One can note that for bosonic ($\alpha = 0$) and fermionic ($\alpha = 1$) cases the eigenvalue λ is two-fold degenerate (except when $l = 0$), while it is not degenerate for any other value of α. By substituting the eigenvalue λ we can now explicit the radial equation

$$\left[-\frac{\hbar^2}{m} \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) + \frac{\hbar^2}{mr^2} (l - \alpha)^2 \right] R(r) = E\, R(r), \tag{1.32}$$

which deserves a few comments.

First of all, the effect ot the anyonic statistics is to replace the angular momentum l by $l - \alpha$ in the radial equation for the relative motion. This is true not only for the ideal case, but for any two-anyon problem. So in all cases, but the case of bosons, particles experience a centrifugal barrier.

Equation (1.32) is nothing else than the Bessel equation, whose solution nonsingular at the origin are

$$R(r) = J_{|l-\alpha|}(kr), \quad E = \frac{\hbar^2 k^2}{m} \tag{1.33}$$

and the spectrum is continuous. For $r \to 0$, the nonsingular solution behaves like

$$R(r \to 0) \sim r^{|l-\alpha|} \tag{1.34}$$

So it is clear that in the ground state ($l = 0$) the repulsion bewteen two anyons monotonically increases by moving from $\alpha = 0$ (bosons, which experience no repulsion) to $\alpha = 1$ (fermions, which experience maximun repulsion). In other words, anyons can be regarded as bosons/fermions with an extra repulsive/attractive interaction.

Let us spend instead some words about the interacting case. From the relative Hamiltonian (1.29) it is understood that consideration of a central potential $V \equiv V(r)$ cannot modify the solution (1.31) of the angular equation (which is unchanged), while in the radial equation one had simply to add $V(r)$. An interesting question consists in wondering about which possible forms of $V(r)$ allows for an analytical solution of the corresponding radial equation. Well, the only potentials for which the eigenvalues and the eigenfunctions admit closed analytical form for all the partial waves l, are the oscillator potential and the attractive Coulob potential. Moreover, for the case of repulsive Colomb potential one can write down the partial shifts for all the partial waves. Furthermore, in both the cases we are free to add the potential A/r^2 without affecting the analytical solvability, because the constant A can be reabsorbed in the quantum number l. The solutions for ther cases, such as the case of anyons in a circular box with hard walls, or experiencing hard-disk repulsion, rely on the solution of trascendental equations. A system of two anyons embedded in a uniform magnetic field (with or without the optional presence of an oscillator potential)can be reduced to the oscillator case, so it is exactly solvable too. In general most of the possible central potential are hard to deal with.

Chapter 2

Chern-Simons theory and Anyons

2.1 Chern-Simons density

In this Chapter we will see non-relativistic quantum theories harbouring anyonic statistics. This is possible thanks to the presence of the topological Chern-Simons term in the action. So now we define the Chern-Simons term in $2+1$ dimensions and discuss its properties. A fundamental property of the Chern-Simons term in $2+1$ dimensions is that by adding it to a theory containing the gauge kinetic energy term, it makes massive the gauge field, by still preserving the gauge invariance of the action. After the Abelian theory, we will present the non-Abelian (CS) gauge theories, in which the coefficient of the Chern-Simons term has to be quantized to make them well defined. A gauge theory with pure Chern-Simons action is an example of a topological field theory, since this term has the same form in the flat and the curved space-time, irrespectively of the metric tensor. In a wider context, the Chern-Simons term has found many applications, for instance in condensed matter physics (fractional quantum Hall effect), supergravity, string theory. We will see in particular how the anyonic statistics can be formulated by using the Chern-Simons term.
In order to understand this term, let us start with the Lagrangian density for classical electrodynamics in $3+1$ dimensions:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}\left(i\gamma_\mu D^\mu - m\right)\psi \tag{2.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and $D_\mu = \partial_\mu - ieA_\mu$ is the covariant derivative. A_μ denotes the gauge potential, ψ the fermionic field. The gauge invariance of the Lagrangian is expressed by the gauge transformation

$$\begin{cases} \psi(x) \to e^{ie\alpha(x)}\,\psi(x) \\ \\ A_\mu(x) \to A_\mu(x) + \partial_\mu\alpha(x) \end{cases} \tag{2.2}$$

In the case of massless fermion a further symmetry (the **axial symmetry**) is present in this Lagrangian, which is expressed by the invariance under the transformation

$$\begin{cases} \psi(x) \to e^{i\beta\gamma_5}\,\psi(x) \\ \\ A_\mu(x) \to A_\mu(x) \end{cases} \tag{2.3}$$

where $\beta \in R$ and $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$. Indeed for $m = 0$ the Lagrangian can be written as

$$\mathcal{L}_{m=0} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\overline{\psi}_R\,\gamma_\mu D^\mu\,\psi_R + i\overline{\psi}_L\,\gamma_\mu D^\mu\,\psi_L, \tag{2.4}$$

where the *chiral* fields $\psi_R(x), \psi_L(x)$ evidently do not couple each other. Consequently the Lagrangian is seen to be invariant under a (global) chiral phase change named $U(1)_R \times U(1)_L$

$$\begin{cases} \psi_L(x) \to e^{-i\theta_L}\psi_L(x) \\ \\ \psi_R(x) \to e^{-i\theta_R}\psi_R(x) \end{cases} \tag{2.5}$$

with θ_R, θ_L real constant phases. Applying the Noether theorem, two conserved currents are produced (i.e. $\partial_\mu J^\mu_L = 0$ and $\partial_\mu J^\mu_R = 0$)

$$\begin{cases} J^\mu_L = \overline{\psi}_L\gamma^\mu\psi_L \\ \\ J^\mu_R = \overline{\psi}_R\gamma^\mu\psi_R \end{cases} \tag{2.6}$$

Now the axial symmetry comes in the following way; two conserved currents can be obtained as linear combinations of the currents above:

$$\mathcal{V}^\mu = J^\mu_R + J^\mu_L$$

$$\mathcal{A}^\mu = J^\mu_R - J^\mu_L,$$

which are respectively a vector current

$$\mathcal{V}^\mu = \overline{\psi}\gamma^\mu\psi$$

and an axial-vector current

$$\mathcal{A}^\mu = \overline{\psi}\gamma^\mu\gamma_5\psi$$

Associated with this new symmetry group $U(1)_V \times U(1)_A$, we have the two transformations

$$\mathcal{V} : \psi(x) \rightarrow e^{-i\theta_V}\psi(x); \quad \mathcal{A} : \psi(x) \rightarrow e^{-i\theta_A\gamma_5}\psi(x) \tag{2.7}$$

where θ_V, θ_A are (global) constants, the gauge potential field $\psi(x)$ is let invariant. The first of them is a particular case of the gauge invariance, the global gauge invariance. The second one is just the desired *axial symmetry* stated in (2.3).
These two symmetries, the gauge and the chiral symmetries, are valid (and their respective currents $j_\mu = \overline{\psi}\gamma^\mu\psi$ and $j^5_\mu = \overline{\psi}\gamma_\mu\gamma_5\psi$ are conserved) at the classical level, but they are **not** conserved at the quantum level [21]. The problem in the quantum case resides in the fact that the fundamental quantization condition for Dirac-Fermi fields

$$\psi^\dagger_m(t,\mathbf{r})\,\psi_n(t,\mathbf{r'}) + \psi_n(t,\mathbf{r'})\,\psi^\dagger_m(t,\mathbf{r}) = \delta_{mn}\delta^3(\mathbf{r}-\mathbf{r'}) \tag{2.8}$$

implies that the product of $\psi^\dagger$ and ψ at the same space-time point is necessarily singular. [In the above (m,n) labels the components of $\psi^\dagger$ and ψ]. Since the charges and currents involve bilinears of the Dirac-Fermi fields at the same space-time point, they are necessarily ill-defined in the quantum theory. Hence regularization and renormalization are needed to render the currents well-defined. But it turns out that any regularization/renormalization method in the presence of the vector field A violates the symmetries that are present in the unquantized theory. It is possible to preserve one of the two corrents (or a linear combination of the two) but not both. Since the preservation of both symmetries is impossible, a choice must be made about which one should be preserved. Since local gauge symmetries are frequently needed for consistency reasons, they are the ones that are preserved, while global axial gauge symmetries are abandoned: they become affected by "anomalies". Indeed, when discovered, this phenomenon was baptized quantum chiral *anomaly*, since unexpected, and it is a paradigmatic example of quantum mechanical symmetry breaking. This breaking of the (classical) chiral symmetry is valid in any even dimension $2n$, in both Abelian and non-Abelian gauge theories. The symmetry breaking ("deviation from the conservation", or simply divergence) for the chiral current can be encoded in the **Chern -Pontryagin** density P_{2n} in that even dimension $2n$

$$\partial^\mu j^5_\mu \propto P_{2n} \tag{2.9}$$

The Chern-Pontryagin density can be written as a total divergence:

$$P_{2n} = \partial_\mu \Gamma^\mu, \quad \mu = 0, 1, 2, \cdots, 2n-1. \tag{2.10}$$

The components of the contravariant vector Γ^μ live in odd $(2n-1)$ dimensions, and this vector is called the **Chern-Simons** density in $(2n-1)$ dimensions. So the Chern-Pontryagin density lives in an even space-time dimensionality, while the Chern-Simons density has odd dimensionality. In our basic example represented by the classical electrodynamics in $3+1$ dimensions, the Chern-Simons density takes the form

$$\partial^\mu j^5_\mu = \frac{e^2}{2\pi}\epsilon_{\mu\nu\lambda\sigma}F^{\mu\nu}F^{\lambda\sigma} = \frac{e^2}{\pi}\partial^\mu(\epsilon_{\mu\nu\lambda\sigma}A^\nu F^{\lambda\sigma}), \tag{2.11}$$

whence the Abelian Chern-Simons term in $2+1$ dimensions is expressed by

$$J_{CS} = \int \mathcal{L}_{CS}\, d^3x \propto \int d^3x\, \epsilon_{\nu\lambda\sigma}A^\nu F^{\lambda\sigma}. \tag{2.12}$$

2.2 Gauge invariant mass term

Let's take the pure electrodynamics in the presence of the Chern-Simons term in $2+1$ dimensions [22, 23]

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\mu}{4}\epsilon^{\mu\nu\lambda}F_{\mu\nu}A_\lambda \tag{2.13}$$

where, as prescribed by dimensional analysis, μ has the dimension of a mass in $2+1$ dimensions. The equation of motion following after this Lagrangian is

$$\partial_\mu F^{\mu\nu} + \frac{\mu}{2}\epsilon^{\nu\alpha\beta}F_{\alpha\beta} = 0 \tag{2.14}$$

which is invariant under the gauge transformation $A_\mu \to A_\mu + \partial_\mu\alpha$. The resulting Lagrangian density takes a total derivative as an addendum, and the corresponding total action is gauge invariant. We deduce that the Chern-Simons term (at least the non-Abelian one) has nontrivial topology and also nontrivial dynamics. We can rewrite the equation of motion above in the form

$$\left(g^{\mu\nu} + \frac{1}{\mu}\epsilon^{\mu\nu\alpha}\partial_\alpha\right){}^*F_\nu = 0, \tag{2.15}$$

where ${}^*F_\nu$ is the dual field strength, which is a vector in $2+1$ defined as follows

$${}^*F_\nu \equiv \frac{1}{2}\epsilon_{\nu\alpha\beta}F^{\alpha\beta}; \quad F_{\mu\nu} = \epsilon_{\mu\nu\alpha}{}^*F^\alpha. \tag{2.16}$$

Now the arising of mass for the gauge field will be explained. Let's act with the left operator $(g_{\beta\eta} - \frac{1}{\mu}\,\epsilon_{\beta\eta\delta}\,\partial^\delta)$ on Eq.(2.15), getting as a result

$$(\Box + \mu^2)^* F_\beta = 0. \tag{2.17}$$

From last equation it is clear that the gauge field excitations are **massive**, and the mass μ of the gauge field is just the coefficient of the Chern-Simons term. Summarizing, when adding the Chern-Simons term to the Maxwell kinetic energy term for the gauge field, we end up with a gauge invariant mass term for the gauge field. This is a special property of the (2+1)-dimensional case. Since we made a comment about the dynamics, it is the occasion to point out here that a **pure** Chern-Simons action (considered alone without a kinetic energy term) has **no** dynamics at all: its equation of motion is $F_{\mu\nu} = 0$. That means that, while the Chern-Simons gauge field taken alone is a non-propagating field, its dynamics is completely inherited from the fields to which it is eventually minimally coupled. Indeeed if one considers the Lagrangian

$$\mathcal{L} = \frac{\mu}{4}\epsilon_{\mu\nu\lambda}F^{\mu\nu}A^\lambda + A_\mu J^\mu \tag{2.18}$$

in which the first term is nothing but the pure Chern-Simons Lagrangian while the second is the coupling of the Chern-Simons field with a current, function of some other fields, the resulting equations of motion are

$$\rho \equiv J^0 = \mu B \tag{2.19}$$

$$J^i = \mu\epsilon^{ij}E_j. \tag{2.20}$$

The first is a Gauss law constraint, implying a relation between the Noether charge Q and the total flux Φ:

$$Q \equiv \int \rho\, d^2x = \mu \int B d^2x \equiv \mu\Phi. \tag{2.21}$$

A property of the pure Chern-Simons term is that the action corresponding to it contains only up to the first order in the time derivative. So the components A_1, A_2 of the gauge field are canonically conjugate each other. In the next Chapter we will come to consider the Abelian *pure* Chern-Simons theory, with the purpose of discussing many relations between the observables of the theory (which are expectation values of Wilson line operators) and the homological properties of the particles' closed worldlines.

2.3 Non-Abelian case

Relations among quantities analogous to those in Sec. 2.1 can be written for the non-Abelian case, leading to a modified expression for the Chern-Simons Lagrangian density [24, 25]. The non-Abelian case is obtained by replacing electrodynamics with a more general Yang-Mills theory. To this end, we replace the function A_μ by a matrix-valued quantity $A_\mu \equiv \sum_a A_\mu^a T_a$ living in a Lie algebra, where T_a are anti-Hermitian representation matrices satisfying the Lie algebra commutators with structure constraints f_{ab}^c:

$$[T_a, T_b] = \sum_c f_{ab}^c T_c \tag{2.22}$$

and normalized by Tr $T_a T_b = -\delta_{ab}/2$. For $SU(2)$, $T_a = \sigma_a/2i$, σ_a Pauli matrices. The singlet axial vector current J_5^μ obeys the anomalous continuity equation

$$\frac{\partial}{\partial x^\mu} J_5^\mu(x) = \frac{e^2}{\pi} \mathrm{Tr}[{}^*F^{\mu\nu}(x) F_{\mu\nu}(x)], \tag{2.23}$$

where $F_{\mu\nu}$ is the non-Abelian field strength constructed from A_μ (Yang-Mills curvature):

$$F_{\mu\nu} \equiv \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + [A_\mu(x), A_\nu(x)] \tag{2.24}$$

and ${}^*F^{\mu\nu}$ is its dual:

$${}^*F^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}. \tag{2.25}$$

In analogy with the Abelian case, the term on the r.h.s. of the (2.23) is the Pontryagin density P:

$$P \equiv -\frac{e^2}{2\pi} \mathrm{Tr}[{}^*F^{\mu\nu}(x) F_{\mu\nu}(x)], \tag{2.26}$$

whose 4-dimensional integral measures the topological properties of the Yang-Mills gauge potential A_μ (connections) and fields $F_{\mu\nu}$ (curvatures) that enter in P. For the integral to converge, $F_{\mu\nu}$ must tend to zero at infinite argument. This means that A_μ must tend to a pure gauge g, which is group valued,

$$A_\mu(x) \to g^{-1}(x) \frac{\partial}{\partial x^\mu} g(x) \tag{2.27}$$

and g is restricted to tend to the identity. Gauge functions g with this restriction fall into the equivalence (homotopy) class labeled by integers, and gauge functions in different classes cannot be deformed into each other. That integer n is given by the Pontryagin number

$$n = \int d^4x\, P(x). \tag{2.28}$$

The Pontryagin density P is a gauge invariant object, and equal to the 4-divergence $P(x) = \frac{\partial}{\partial x^\mu} K^\mu(x)$ of the contravariant vector Chern-Simons density (living in 3 dimensions) K^μ , also called topological current

$$K^\mu(x) = -\frac{e^2}{2\pi}\epsilon^{\mu\alpha\beta\gamma}\,\mathrm{Tr}\left[\frac{1}{2}A_\alpha(x)\,\partial_\beta A_\gamma(x) + \frac{1}{3}A_\alpha(x)A_\beta(x)A_\gamma(x)\right], \quad (2.29)$$

where $A_\mu \equiv T^a A^a_\mu$. The 4-dimensional volume integral of ${}^*F^{\mu\nu}(x)F_{\mu\nu}(x)$ in the integral formula for the Pontryagin number can be clearly written as an integral of K^μ over the 3-dimensional surface (at infinity) bounding the 4-dimensional volume. There the vector potentials in K^μ are replaced by their asymptotic forms given by a pure gauge, and the resulting integration gives the integer n that characterizes the winding number of g. The Pontryagin number is a topological entity, indeed as just emphasized it is determined by the asymptotic behavior of gauge functions, which belong to distinct classes labeled by integers. From a different perspective, one can also argue that the integral of the Potryagin density above addressed does not require to specify the geometry of the integration 4-volume, because the metric tensor does not appear in the formula. After having described the arising of the non-Abelian Chern-Simons term, we can consider a non-Abelian gauge theory with the Chern-Simons term as given by

$$\mathcal{L}_{na} = \frac{1}{2g^2}\mathrm{Tr}\,F^{\mu\nu}F_{\mu\nu} - \frac{\mu}{2g^2}\,\epsilon^{\mu\nu\lambda}\,\mathrm{Tr}\left[F_{\mu\nu}A_\lambda - \frac{2}{3}A_\mu(x)A_\nu(x)A_\gamma(x)\right], \quad (2.30)$$

where now $A_\mu \equiv gT^a A^a_\mu$ and $F_{\mu\nu} \equiv gT^a F^a_{\mu\nu}$, consistently with Eq.(2.24). This Lagrangian yields the following field equation

$$D_\mu F^{\mu\nu} + \frac{\mu}{2}\epsilon^{\nu\alpha\beta}F_{\alpha\beta} = 0, \quad (2.31)$$

where

$$D_\mu \equiv \partial_\mu + [A_\mu,]$$

is gauge covariant. As in the Abelian case, the Chern-Simons term induces a gauge invariant gauge field mass μ. By remembering the dual strength field defined in the previous Section, the non-Abelian version of (2.17) is

$$(D_\nu D^\nu + \mu^2)\,{}^*F_\lambda = \epsilon_{\lambda\delta\eta}\,[{}^*F_\delta, {}^*F_\eta] \quad (2.32)$$

Also in the non-Abelian case, the Chern-Simons Lagrangian density changes by a total derivative under an infinitesimal local gauge transformation, so that the corresponding action is invariant under such a gauge transformation; but the action is instead **not** invariant under finite gauge transformations!: in fact, the gauge transformations not continuously deformable to the identity, i.e. homotopically non-trivial, change the action as follows. Let

$$A_\mu \to U^{-1} A_\mu U + U^{-1} \partial_\mu U \tag{2.33}$$

be the gauge transformation. In the case of gauge group $SU(2)$, the variation of the action under such "large" gauge transformations will be

$$S_{na} \to S_{na} + \frac{8\pi^2 \mu}{g^2} \omega(U), \tag{2.34}$$

where

$$\omega(U) = \frac{1}{24\pi^2} \int d^3x\, \epsilon^{\mu\nu\lambda}\ \mathrm{Tr}\left[(\partial_\mu U) U^{-1} (\partial_\nu U) U^{-1} (\partial_\lambda U) U^{-1} \right] \tag{2.35}$$

is the winding number of the gauge transformation U [26]. This is true for any gauge group G of which $SU(2)$ is a sub-group. For these groups, the value of $\omega(U)$ is an integer for any U, so the action transforms as

$$S_{na} \to S_{na} + \frac{8\pi^2 \mu}{g^2} m, \tag{2.36}$$

where m is an integer. To sum up, the action corresponding to the non-Abelian gauge theory with the Chern-Simons term is not invariant under large gauge transformations, but it rather changes by $8\pi^2 \mu m / g^2$. In the path-integral formulation, the physical requirement is not the gauge invariance of the action, while rather that of the quantity $\exp\{iS_{na}\}$. So the theory itself is meaningful if and only if the Chern-Simons mass μ is **quantized** as an integer multiple of $g^2/4\pi$. A final remark concerning the Abelian as well the non-Abelian case: the Chern-Simons action depends only on the anti-symmetric tensor $\epsilon_{\mu\nu\lambda}$ and not on the metric tensor, this meaning that it is the same in the flat and the curved space. Hence it is a particular topological field theory [27, 44, 28]. The topological field theories are a natural framework for studying the Jones polynomials, arising in knot theory, by dealing with three dimensional terms.

2.4 Non-Abelian Chern-Simons particles

The non-Abelian Chern-Simons particles (NACS particles) are point-like sources carrying non-Abelian charges, and interacting via the non-Abelian

Chern-Simons term. Their interaction produces the non-Abelian Aharonov-Bohm effect, as well as the Aharonov-Bohm effect is experienced by Abelian anyons due to their relative interaction. The NACS particles can occur in various contexts, such as cosmic strings [29], gravitational scattering in (2+1) dimensions [30], and potentially in the ambit of topological insulators [31]. These (quasi-)particle are non-Abelian generalizations of anyons. A difference is represented by a fractionalization of their statistis: while Abelian anyons have fractional spins and satisfy anyon statistics, the NACS particles acquire fractional but rational spins and exhibit generalized braid non-Abelian statistics. The NACS particles also appear in the vortices, in $(2+1)$ dimensions, formed when a gauge group is broken via Higgs mechanics to a discrete non-Abelian subgroup [32]. Interactions among these vortices are expressed in terms of holonomies, associated with the windings around themselves, so these interactions are a manifestation of the non-Abelian Aharonov-Bohm effect.

We will present the definition of these particles by endowing point-like sources, having non-Abelian isospin charges, with non-Abelian magnetic fluxes. This will be obtained by introducing the NACS term and minimally coupling the isospin charges with the Chern-Simons gauge fields. For simplicity the internal symmetry group will be assumed to be $SU(2)$. Let us define the isospin degrees of freedom, on the reduced space phase S^2 for the $SU(2)$ internal symmetry group, as

$$Q^1_\alpha = J_\alpha \sin\theta_\alpha \cos\phi_\alpha, \quad Q^2_\alpha = J_\alpha \sin\theta_\alpha \sin\phi_\alpha, \quad Q^3_\alpha = J_\alpha \cos\theta_\alpha, \tag{2.37}$$

$\theta_\alpha, \phi_\alpha$ coordinates of the internal S^2 group, J_α constant. We are going to denote the spatial coordinates of the N particles by q_α, $\alpha = 1, , 2, \cdots, N$ in the following classical Lagrangian:

$$L = \sum_\alpha (-\frac{1}{2} m_\alpha \dot{q}^2_\alpha + J_\alpha \cos\theta_\alpha \dot{\phi}_\alpha) - \kappa \int d^2x\, \epsilon^{\mu\nu\lambda}\, \mathrm{Tr}(A_\mu \partial_\nu A_\lambda + \frac{2}{3} A_\mu A_\nu A_\lambda)$$

$$+ \int d^2x \sum_\alpha (A^a_i(t,x)\dot{q}^i_\alpha + A^a_0(t,x))\, Q^a_\alpha\, \delta(x - q_\alpha). \tag{2.38}$$

In this classical Lagrangian $4\pi\kappa$ is an integer, $A_\mu = A^a_\mu T^a$, $[T^a, T^b] = \epsilon^{abc} T^c$, $\mathrm{Tr}(T^a T^b) = -\frac{1}{2}\delta_{ab}$, and the space-time signature is $(+, -, -)$. The corresponding Euler-Lagrange equations are then

$$m_\alpha \ddot{q}_{\alpha i} = -(F^a_{ij}(q_\alpha)\dot{q}^j_\alpha + F^a_{i0}(q_\alpha))\, Q^a_\alpha \tag{2.39}$$

$$\dot{Q}^a_\alpha = -\epsilon^{abc}(A^b_i(q_\alpha)\dot{q}^i_\alpha + A^b_0(q_\alpha))\, Q^c_\alpha \tag{2.40}$$

$$\frac{\kappa}{2}\epsilon^{ij}F^a_{ij}(x) = -\sum_\alpha Q^a_\alpha \delta(x-q_\alpha) \tag{2.41}$$

$$\kappa\,\epsilon^{ij}F^a_{j0}(x) = -\sum_\alpha Q^a_\alpha\,\dot{q}^i_\alpha\,\delta(x-q_\alpha), \tag{2.42}$$

where $F^a_{ij} = \partial_i A^a_j - \partial_j A^a_i + \epsilon^{abc}\,A^b_i\,A^c_j$. Within this set of four equations of motion, the first two are the Wong's equations [33], and the third one is the Gauss' law constraint, telling that the NACS particle of isospin charge Q^a_α carries the magnetic flux $-Q^a_\alpha/\kappa$,

$$\Phi_m = \frac{1}{2}\int_{B_\alpha}\epsilon^{ij}F^a_{ij}(x)\,d^2x = -\frac{1}{\kappa}Q^a_\alpha, \tag{2.43}$$

where B_α is a surface including the position (only) of the α-th particle.

Introducing the canonical momenta p^i_α

$$p^i_\alpha = \frac{\partial L}{\partial \dot{q}_{i\alpha}} = m_\alpha \dot{q}^i_\alpha + A^{ai}(q_\alpha)\,Q^a_\alpha \tag{2.44}$$

the Lagrangian can be rewritten in a first-order form as we will use in the following. At this point is useful recalling that A^a_1 and A^a_2 are canonically conjugates to each other, so their quantization will have commutation rules:

$$[\hat{A}^a_1(x), \hat{A}^a_2(y)] = i\kappa^{-1}\,\delta^{ab}\,\delta(x-y). \tag{2.45}$$

We are going to summarize the steps of the coherent states quantization [34, 35] of this Lagrangian. The commutation rules among the components of the gauge field suggest to consider "creation" and "annihilation" operators

$$\mathcal{A}^{a\dagger} = \sqrt{\kappa/2}(\hat{A}^a_1 - i\,\hat{A}^a_2), \quad \mathcal{A}^a = \sqrt{\kappa/2}(\hat{A}^a_1 + i\,\hat{A}^a_2) \tag{2.46}$$

$$[\mathcal{A}^a(x), \mathcal{A}^{a\dagger}(y)] = \delta^{ab}\delta(x-y) \tag{2.47}$$

and to construct coherent states

$$|A_{\bar{z}}\rangle \equiv \exp\left(\sqrt{\kappa/2}\int d^2x\,A^a_{\bar{z}}\mathcal{A}^{a\dagger}\right)|0\rangle \tag{2.48}$$

with their adjoints

$$\langle A_z| \equiv \langle 0|\exp\left(\sqrt{\kappa/2}\int d^2x\,A^a_z\mathcal{A}^a\right). \tag{2.49}$$

Hence

$$\langle A_z|A_{\bar{z}}\rangle = \exp\left(\frac{\kappa}{2}\int d^2x A^a_z A^a_{\bar{z}}\right), \tag{2.50}$$

yielding the following decomposition for the identity

$$I = \int DA_z\, DA_{\bar z}\, \exp\left(-\frac{\kappa}{2}\int d^2x\, A^a_z A^a_{\bar z}\right)\, |A_{\bar z}\rangle\langle A_z| \tag{2.51}$$

The physical transition amplitude between gauge fields, $Z \equiv \langle A^f_z, t_f | A^i_{\bar z}, t_i\rangle$, can be expressed via the functional integral representation, and with the aid of the identity representation written above, in the following form

$$Z = \int Dp^z\, Dq^{\bar z}\, Dp^{\bar z}\, Dq^z\, D\cos\theta\, D\phi\, DA_z\, DA_{\bar z}\, DA_0$$

$$\times \exp\left\{-\kappa\, i \int d^2z (A^f_{\bar z}A^f_z + A^i_{\bar z}A^i_z)\right\} \exp\left\{i\int dt\, L\right\}, \tag{2.52}$$

where

$$L = \sum_\alpha (p^{\bar z}_\alpha \dot z_\alpha + p^z_\alpha \dot{\bar z}_\alpha + J_\alpha\, \cos\theta_\alpha \dot\phi_\alpha)$$

$$+ \int d^2z\, \left(\frac{\kappa}{2}(\dot A^a_z A^a_{\bar z} - \dot A^a_{\bar z} A^a_z) + A^a_0 \Phi^a\right) - H, \tag{2.53}$$

the Hamiltonian is

$$H = \sum_\alpha \frac{2}{m_\alpha}\, (p^{\bar z}_\alpha - A^a_z(z_\alpha, \bar z_\alpha)\, Q^a_\alpha)\, (p^z_\alpha - A^a_{\bar z}(z_\alpha, \bar z_\alpha)\, Q^a_\alpha)\,, \tag{2.54}$$

and $\Phi^a(z) = \kappa F^a_{z\bar z} + \sum_\alpha Q^a_\alpha \delta(z - z_\alpha) = 0$. Since the enlarging of the gauge orbit space, due to the fact that A^a_z and $A^a_{\bar z}$ are treated as independent variables within the frame of the coherent state quantization, we are free to choose $A^a_{\bar z} = 0$ as a gauge fixing condition, that can be called "holomorphic" gauge condition. In this gauge the Gauss' constraint takes the form

$$\Phi^a(z) = -\kappa\, \partial_{\bar z} A^a_z + \sum_\alpha Q^a_\alpha \delta(z - z_\alpha) = 0 \tag{2.55}$$

explicitly solved by

$$A^a_{\bar z}(z, \bar z) = 0, \quad A^a_z(z, \bar z) = \frac{i}{2\pi\kappa}\sum_\alpha Q^a_\alpha \frac{1}{z - z_\alpha}. \tag{2.56}$$

The legitimacy of the holomorphic gauge condition as a gauge fixing condition is guaranteed by the Fradkin and Vilkovisky theorem [36], stating the equivalence of the path integral in the holomorphic gauge condition to those in conventional gauge choices.

In the coherent state quantization for the Chern-Simons gauge fields, we are left [37] with the following path integral for the physical transition amplitude

$$Z = \int Dp^z\, Dq^{\bar{z}}\, Dp^{\bar{z}}\, Dq^z\, D\cos\theta\, D\phi\, DA_z\, DA_{\bar{z}}\, \delta(A^a_{\bar{z}})\delta(\Phi^a)\, \exp\left\{i \int dt\,(K - H)\right\},$$

$$K = \sum_\alpha (p^{\bar{z}}_\alpha \dot{z}_\alpha + p^z_\alpha \dot{\bar{z}}_\alpha + J_\alpha\, \cos\theta_\alpha \dot{\phi}_\alpha) + \int d^2z\, \frac{\kappa}{2}(\dot{A}^a_z A^a_{\bar{z}} - \dot{A}^a_{\bar{z}} A^a_z) \qquad (2.57)$$

A **quantum mechanical** description of the NACS particles can be obtained by integrating out the field variables. The physical transition amplitude Z can be expressed in terms of purely quantum mechanical variables, by using the solution (2.56) of the holomorphic gauge condition:

$$Z = \int Dp^z\, Dq^{\bar{z}}\, Dp^{\bar{z}}\, Dq^z\, D\cos\theta\, D\phi\, \exp\left\{i \int dt\,(K - H)\right\},$$

$$K = \sum_\alpha (p^{\bar{z}}_\alpha \dot{z}_\alpha + p^z_\alpha \dot{\bar{z}}_\alpha + J_\alpha\, \cos\theta_\alpha \dot{\phi}_\alpha),$$

$$H = \sum_\alpha \frac{2}{m_\alpha}\, p^z_\alpha\, (p^{\bar{z}}_\alpha - A^a_z(z_\alpha, \bar{z}_\alpha) Q^a_\alpha)\,. \qquad (2.58)$$

It is equivalent to the following operatorial version

$$Z = \langle \eta_f | \exp\{-i\, \hat{H}\, (t_f - t_i)\} | \eta_i \rangle$$

$$\hat{H} = \sum_\alpha \frac{2}{m_\alpha}\, \hat{p}^z_\alpha\, (\hat{p}^{\bar{z}}_\alpha - \hat{A}^a_z(z_\alpha, \bar{z}_\alpha) \hat{Q}^a_\alpha), \qquad (2.59)$$

where the gauge field $\hat{A}^a_z$ is the operator version of the solution field in (2.56), and the quantum operator variables involved in the expression fulfill

$$[\bar{z}_\alpha, \hat{p}^z_\alpha] = i \quad [z_\alpha, \hat{p}^{\bar{z}}_\alpha] = i \quad [\hat{Q}^a_\alpha, \hat{Q}^b_\beta] = i\, \epsilon^{abc} \hat{Q}^c_\alpha\, \delta_{\alpha\beta} \qquad (2.60)$$

The final expression governing the non-relativistic dynamics of the NACS particles is the Hamiltonian

$$\hat{H} = -\sum_\alpha \frac{1}{m_\alpha} (\nabla_{\bar{z}_\alpha} \nabla_{z_\alpha} + \nabla_{z_\alpha} \nabla_{\bar{z}_\alpha}), \qquad (2.61)$$

where

$$\nabla_{z_\alpha} = \partial / \partial z_\alpha + \frac{1}{2\pi\kappa} \sum_{\beta \neq \alpha} \hat{Q}^a_\alpha \hat{Q}^a_\beta \frac{1}{z_\alpha - z_\beta},$$

$$\nabla_{\bar{z}_\alpha} = \partial/\partial \bar{z}_\alpha \,.$$

This Hamiltonian has been applied to the non-Abelian Aharonov-Bohm effect by Verlinde [38]. The term proportional to κ^{-1} in the covariant derivative describes mutual interactions among NACS particles, which are responsible for the non-Abelian statistics. This fact can be proven by applying the following singular non-unitary transformation U which connects the holomorphic gauge with the anyon gauge (similarly to what previously illustrated in the case of (Abelian) anyons):

$$\Psi_h(z_1, \cdots, z_N) = U^{-1}(z_1, \cdots, z_N)\, \Psi_a(z_1, \cdots, z_N), \tag{2.62}$$

and satisfies the Knizhnik-Zamolodchikov equation [39]

$$\left(\frac{\partial}{\partial z_\alpha} + \frac{1}{2\pi\kappa} \sum_{\beta \neq \alpha} \hat{Q}^a_\alpha \hat{Q}^a_\beta \frac{1}{z_\alpha - z_\beta} \right) U^{-1}(z_1, \cdots, z_N) = 0 \,. \tag{2.63}$$

The same transformation applied to the Hamiltonian produces a **free** Hamiltonian in the anyon basis:

$$\hat{H}_a = -\sum_\alpha \frac{2}{m_\alpha} \partial_{\bar{z}_\alpha \partial_{z_\alpha}} . \tag{2.64}$$

The Knizhnik-Zamolodchikov equation satisfied by the U matrix is particularly simple for the case of $N = 2$ particles, so that this monodromy can be explicitly evaluated in this case:

$$U(z_1, z_2) = \exp \left[\hat{Q}^a_1 \hat{Q}^a_2 \, \frac{1}{2\pi\kappa} \, \ln(z_1 - z_2) \right] \tag{2.65}$$

As a result, the exchange of the positions of two NACS along an oriented path produces the following UNITARY wave function transformation in the anyon gauge

$$\Psi_a(z_1, z_2) \to \Psi_a(z_2, z_1) = e^{\left(i \frac{\hat{Q}^a_1 \hat{Q}^a_2}{2\kappa} \right)} \, \Psi_a(z_1, z_2) \tag{2.66}$$

The unitary operator $\mathcal{R}_{\alpha\beta} \equiv \exp \left(i \, \hat{Q}^a_1 \hat{Q}^a_2 / 2\kappa \right)$ above is the **braid operator**, satisfying the Yang-Baxter equation and qualifying the **NACS particles as an example of non-Abelian anyons**. Naturally, the monodromy operator $\mathcal{M}_{\alpha\beta} = (\mathcal{R}_{\alpha\beta})^2$, corresponding to the total winding of a particle around another, produces the transformation

$$\Psi_a(z_1, z_2) \to e^{\left(i \frac{\hat{Q}^a_1 \hat{Q}^a_2}{\kappa} \right)} \, \Psi_a(z_1, z_2) \,. \tag{2.67}$$

Chapter 3

Abelian link invariants and homology

3.1 Introduction

Quantum field theories can be used not only to describe the physics of elementary particles but also to compute topological invariants. In this Chapter we consider the link invariants that are defined by the abelian $U(1)$ Chern-Simons field theory formulated in a closed and oriented 3-manifold M, and we show how these invariants are related with the homology group of the complement of the links. To this end, we shall introduce a few definitions —like that of simplicial satellite or of equivalent knot— which are used to connect the values of the $U(1)$-charges which are associated with the components of the links with the numbers that classify the homology classes of loops.

We demonstrate that the set of the abelian link invariants (or observables) in any homology sphere coincides with the set of observables in the sphere S^3. We also prove that if a link in a generic manifold M is homologically trivial then its associated observable coincides with an observable computed in S^3. We then consider the $U(1)$ surgery invariant of Reshetikhin-Turaev, we show that this invariant: (1) is trivial for homology spheres, (2) is not a function of the homology group of the manifold only and (3) is not a function of the homotopy type of the manifold only.

3.2 Introduction to the Singular Homology

This Section is devoted just to recall the mathematical setting required to understand the rest of the Chapter, relevant to the connection between the first homology group and some invariants associated to the Abelian Chern-Simons theory. I want to give a short account of the concept of homology, by limiting myself to the approach of singular homology [40]. At last, we shall specify the relation between the homology group and the fundamental group (or first homotopy group).

Definition The standard simplex of dimension n, or $n-$*simplex*, is the following subset of R^{n+1} :

$$\Delta_n = \left\{ (x_0, \cdots, x_n) \in R^{n+1} \mid \sum_{i=0}^{n} x_i = 1; x_i \geq 0, i = 0, 1, \cdots, n \right\}. \quad (3.1)$$

The points $v_0 = (1, 0, \cdots, 0), v_1 = (0, 1, 0, \cdots, 0), \cdots, v_n = (0, 0, \cdots, 0, 1)$ are called *vertex* of the simplex.

From this definition it follows that Δ_0 is a single point, Δ_1 is a segment of (straight) line, Δ_2 is a triangular region and Δ_3 is a solid tetrahedron.

Definition A *singular n-simplex* in a topological space X is a continuous function $\phi : \Delta_n \to X$.

As a consequence, a singular $0-$simplex is a point of X, while a singular $1-$simplex is a path in X.

Definition A *singular n-chain* in X is a formal expression

$$\sum_{j \in J} n_j \phi_j$$

where $\{\phi_j | j \in J\}$ is the family of all the singular n-simplex in $X($ J is a set of indices), $n_j \in Z$, and the number of non-zero elements of $\{n_j | j \in J\}$ is finite.

The set $S_n(X)$ of the singular $n-$chains in X forms an Abelian group with respect to the operation defined (in additive notation) by

$$\sum_{j \in J} n_j \phi_j + \sum_{j \in J} m_j \phi_j = \sum_{j \in J} (n_j + m_j) \phi_j.$$

The neutral element is $\sum 0\phi_j$ and the opposite of $\sum_{j \in J} n_j \phi_j$ is $\sum_{j \in J} (-n_j) \phi_j$. Such an operation is associative and the resulting group is Abelian.

The group $S_n(X)$ contains a great deal of elements, that makes difficult its study. Therefore to simplify the treatment we shall introduce the **boundary** operator (in order to define an equivalence relation on the group $S_n(X)$).

If ϕ is a singular $n-$simplex and $i \in \{0, 1, \cdots, n\}$, we define a singular $(n-1)-$simplex $\partial_i \phi$ by posing

$$\partial_i \phi(x_0, x_1, \cdots, x_{n-1}) = \phi(x_0, x_1, \cdots, x_{i-1}, 0, x_{i+1}, \cdots x_{n-1}). \quad (3.2)$$

The extension of the definition of "boundary" to $S_n(X)$ yields a group homomorphism

$$\partial_i : \begin{array}{ccc} S_n(X) & \to & S_{n-1}(X) \\ \sum_{j\in J} n_j\phi_j & \mapsto & \sum_{j\in J} n_j\partial_i\phi_j \end{array} \tag{3.3}$$

Definition The *boundary operator* $\partial : S_n(X) \to S_{n-1}(X)$ is defined as

$$\partial = \partial_0 - \partial_1 + \partial_2 - \cdots + (-1)^n\partial_n = \sum_{i=0}^{n}(-1)^i\partial_i \tag{3.4}$$

The boundary operator allows us to define two remarkable subgroups of $S_n(X)$:

Definition

(a) A singular $n-$chain $c \in S_n(X)$ is an $n-$**cycle** if $\partial c = 0$; the set of the $n-$cycles of X is denoted $Z_n(X)$;

(b) a singular $n-$chain $d \in S_n(X)$ is an $n-$**boundary** if $d = \partial e$ for same $e \in S_n(X)$; the set of the $n-$boundary is denoted $B_n(X)$.

In other words:

$$Z_n(X) = \ker\{\partial : S_n(X) \to S_{n-1}(X)\},$$
$$B_n(X) = \mathrm{Im}\{\partial : S_{n+1}(X) \to S_n(X)\},$$

so both $Z_n(X)$ and $B_n(X)$ are subgroups of $S_n(X)$.

Notes:

1)$Z_0(X) = S_0(X)$;

2)all $n-$boundaries are $n-$cycles, as a consequence of the following

Theorem

$$\partial\partial = 0.$$

The proof is trivial, and consists just in a direct calculation to verify that $\partial\partial$ vanishes on whichever singular $n-$simplex ϕ.:

$$\partial\partial\phi = \partial\left(\sum_{i=0} n(-1)^i\partial_i\phi\right) = \sum_{j=0}^{n-1}\sum_{i=0}^{n}(-1)^{i+j}\partial_j\partial_i\phi = \cdots = 0 \tag{3.5}$$

The former theorem implies that $B_n(X)$ is a subgroup of $Z_n(X)$, so it's a normal subgroup, since $Z_n(X)$ is Abelian: so we are allowed to consider the quotient group $Z_n(X)/B_n(X)$.

Definition The n***-th homology group*** is defined as

$$H_n(X) \equiv Z_n(X)/B_n(X). \tag{3.6}$$

The element of $H_n(X)$ are *homology classes*, that is equivalence classes with respect to the equivalence relation

$$c \sim c' \Longleftrightarrow c - c' \in B_n(X),$$

where $c, c' \in Z_n(X)$.

If $c \sim c'$ we say that c, c' are *homologous* cycles.

One can prove the following facts:

- If the topological space X is made up of only a point, $H_0(X) \cong Z$, and $H_n(X) = \{0\}$ for each $n > 0$.
- If X is a non-empty connected by arcs then $H_0 \cong Z$
- For each continuous map $f : X \to Y$ between two topological spaces X, Y, let us define

  $$f_\sharp : S_n(X) \to S_n(Y) \tag{3.7}$$

  by means of

  $$f_\sharp \left(\sum_{j \in J} n_j \phi_j \right) = \sum_{j \in J} n_j (f \phi_j). \tag{3.8}$$

  $f_\sharp$ is a group homomorphism. It's possible to shows that a group homomorphism

  $$f_* : H_n(X) \to H_n(Y) \tag{3.9}$$

  exists, defined by

  $$f_*[c] = [f_\sharp(c)] \tag{3.10}$$

  where c is an $n-$cycle in X and $[c]$ its homology class. The homomorphism $f_* : H_n(X) \to H_n(Y)$ is called the *homomorphism induced by* f.

  Now we can formulate the

- **Theorem of homotopic invariance**

  Let $f, g : X \to Y$ be two continuous map. If f and g are homotopic map, then $f_* = g_* : H_n(X) \to H_n(Y)$ for each $n \geq 0$, with the just introduced notation.

  An important consequence of the theorem above is that *spaces with isomorphic homotopy groups have isomorphic homology groups.*

  The following theorem makes clear the relation between the homology group and the fundamental group of a topological space.

- **Theorem**

  For any topological space Y there is an homomorphism

  $$\psi : \pi(Y, y_0) \to H_1(Y);$$

  if Y is connected by arcs, ψ is a surjective homomorphism whose core is the subgroup of the commutators of $\pi(Y, y_0)$; in other words, if Y is connected by arcs, $H_1(Y)$ ***is the abelianization of*** *$\pi(Y, y_0)$**.***

Some simple facts about of first homology group:

- $H_1(S^n) = \{0\} \quad \forall n \neq 1$;
- $H_1(S^1) \cong Z$;
- $H_1((S^1)^n) \cong Z^n$;
- two surfaces S_1 and S_2 are homeomorphic if and only if $H_1(S^1) \cong H_1(S^2)$;
- if X is a topological space with the shape of an "eight", we have $H_1(X) = Z \times Z$;
- if S is a closed surface and S' is the space obtained by removing an open disc from S, then $H_1(S) \cong H_1(S')$;
- if Σ_g is the *standard orientable surface of genus g, its first homology group is $H_1(\Sigma_g) = Z^{2g}$.

  *[Can be understood as the connected sum $T^2 \sharp \cdots \sharp T^2$ of g copies of $T^2 = S^1 \times S^1$].

3.3 Field theory approach

In order to make this Chapter self-contained, we add here a preliminary section containing a description of the main developments in the field theory computations of the link invariants together with a brief description of the Reshetikhin-Turaev surgery rules.

The abelian Chern-Simons theory [27, 41, 42, 43] is a gauge theory defined in terms of a $U(1)$-connection A in a closed oriented 3-manifold M. For each oriented knot $C \subset M$, the corresponding holonomy is given by the integral $\int_C A$ which is invariant under $U(1)$ gauge transformations acting on A.

In the standard field theory formulation of abelian gauge theories, the (classical fields) configuration space locally coincides with the set of 1-forms modulo exact forms, $A \sim A + d\Lambda$. However, if one assumes [44, 45] that a complete set of observables is given by the exponential of the holonomies $\{\exp[2\pi i \int_C A]\}$ which are associated with oriented knots C in M, the invariance group of the observables is actually larger than the standard gauge group. In facts, the observables must be locally defined on the classes of 1-forms modulo forms $\widehat{A}$ with integer periods, $A \sim A + \widehat{A}$, $\int_C \widehat{A} = n \in \mathbb{Z}$. This means that the configuration space is defined in terms of the Deligne-Beilinson cohomology classes [46, 47, 45]. So, we shall now consider the Deligne-Beilinson (DB) formulation of the abelian $U(1)$ Chern-Simons gauge theory.

In order to simplify the notation, the classes belonging to the DB cohomology group of M of degree 1, $H^1_D(M)$, will be denoted by A. Let $H^3_D(M)$ be the space of the DB classes of degree 3. The pairing of the DB cohomology groups, which is called the *-product, defines a natural mapping [48]

$$H^1_D(M) \otimes H^1_D(M) \longrightarrow H^3_D(M) \, . \tag{3.11}$$

The *-product of A with A just corresponds to the abelian Chern-Simons lagrangian [45, 49]

$$A * A \longrightarrow A \wedge dA \, . \tag{3.12}$$

Precisely like the integral of any element of $H^3_D(M)$, the Chern-Simons action

$$S = \int_M A * A \longrightarrow \int_M A \wedge dA \tag{3.13}$$

is defined modulo integers; consequently, the path-integral phase factor

$$\exp\{2\pi i k S\} = \exp\left\{2\pi i k \int_M A * A\right\} \tag{3.14}$$

is well defined when the coupling constant k takes integer values

$$k \in \mathbb{Z} \quad , \quad k \neq 0 \, . \tag{3.15}$$

A modification of the orientation of M is equivalent to the replacement $k \rightarrow -k$. Let us consider a framed, oriented and coloured link $L \subset M$ with N components $\{C_1, C_2, ..., C_N\}$. The colour of each component C_j, with $j = 1, 2, ..., N$, is represented by an integer charge $q_j \in \mathbb{Z}$. The classical expression $W(L)$ of the Wilson line is given by

$$W(L) = \prod_{j=1}^{N} \exp\Big\{2i\pi q_j \int_{C_j} A\Big\} = \exp\Big\{2i\pi \sum_j q_j \int_{C_j} A\Big\} \, . \tag{3.16}$$

Each link component which has colour $q = 0$ can be eliminated, and a modification of the orientation of a link component C is equivalent to a change of the sign of the associated charge q. The observables of the Chern-Simons gauge theory in M are given by the expectation values

$$\langle W(L) \rangle \Big|_M = \frac{\int_M DA \, e^{2\pi i k S} \, W(L)}{\int_M DA \, e^{2\pi i k S}} \; , \tag{3.17}$$

where the path integral should be defined on the DB classes which belong to $H^1_D(M)$. More precisely, the structure of the functional space admits a natural description in terms of the homology groups of M, as indicated by the following exact sequence [50, 51]

$$0 \longrightarrow \Omega^1(M)/\Omega^1_{\mathbb{Z}}(M) \longrightarrow H^1_D(M) \longrightarrow H^2(M) \longrightarrow 0 \; , \tag{3.18}$$

where $\Omega^1(M)$ is the space of 1-forms on M, $\Omega^1_{\mathbb{Z}}(M)$ is the space of closed 1-forms with integer periods on M and $H^p(M)$ is the $(p)^{th}$ integral cohomology group of M. Thus, $H^1_D(M)$ can be understood as an affine bundle over $H^2(M)$, whose fibres have a typical underlying (infinite dimensional) vector space structure given by $\Omega^1(M)/\Omega^1_{\mathbb{Z}}(M)$.

The framing of the link components is used to fix the ambiguities, which appear in the computation of the expectation values (3.17) of the composite Wilson line operators, in such a way to maintain the ambient isotopy invariance of the expectation values [52, 53, 49].

Assuming that expression (3.17) is well defined, one can prove [49] the most important properties of the expectation values: (i) the colour periodicity, (ii) the ambient isopoty invariance and (iii) the validity of the satellite relations. We shall briefly discuss these subjects in section 2.3. When expression (3.17) is well defined, the computation of the observables provides the solution of the Chern-Simons field theory in the manifold M.

3.3.1 Fundamental link invariants

When the 3-manifold M coincides with the 3-sphere S^3, one can compute the expectation values (3.17) by means of (at least) two methods: standard perturbation theory or a nonperturbative path integral computation. Both methods give the same answer.

First method.

Since the topological properties of links in $\mathbb{R}^3$ and in S^3 coincide, let us consider the abelian Chern-Simons theory formulated in $\mathbb{R}^3$. In this case, the

Deligne-Beilinson approach is equal to the standard perturbative formulation of the abelian gauge theories. The direct computation of the observables (3.17) by means of standard perturbation theory [53] gives

$$\langle W(L)\rangle\Big|_{S^3} = \langle W(L)\rangle\Big|_{\mathbb{R}^3} = \exp\Big\{-(2i\pi/4k)\sum_{ij} q_i \mathbb{L}_{ij} q_j\Big\} , \tag{3.19}$$

where the off-diagonal elements of the linking matrix $\mathbb{L}_{ij}$, which is associated with the link L, are given by the linking numbers between the different link components

$$\mathbb{L}_{ij} = \ell k(C_i, C_j) = \ell k(C_j, C_i) \quad , \quad \text{for } i \neq j; \tag{3.20}$$

whereas the diagonal elements of the matrix $\mathbb{L}_{ij}$ correspond to the linking numbers of the link components $\{C_j\}$ with their framings $\{C_{j\mathrm{f}}\}$

$$\mathbb{L}_{jj} = \ell k(C_j, C_{j\mathrm{f}}) = \ell k(C_{j\mathrm{f}}, C_j) \ . \tag{3.21}$$

Second method.

Sequence (3.18) implies that $H^1_D(S^3) \simeq \Omega^1(S^3)/\Omega^1_{\mathbb{Z}}(S^3)$ because $H^2(S^3)$ is trivial. By using the property of translation invariance of the functional measure, which can also be expressed in the form of a Cameron-Martin like formula [56], one can introduce [49] a change of variables in the numerator of (3.17) in such a way to factorize out the value of the partition function, which cancels with the denominator. As a result, one can produce an explicit nonperturbative path-integral computation of the observables (3.17) and one finds [49]

$$\langle W(L)\rangle\Big|_{S^3} = \exp\Big\{-(2i\pi/4k)\sum_{ij} q_i \mathbb{L}_{ij} q_j\Big\} , \tag{3.22}$$

which coincides with expression (3.19).

The observables (3.22), which are ambient isotopy invariants, are called the abelian link invariants. They represent the fundamental invariants because, as we shall see, the value of any other topological invariant of the abelian Chern-Simons theory in a generic 3-manifold M can be derived from expression (3.22).

3.3.2 Observables computation

When the Chern-Simons field theory is defined in a nontrivial manifold M, the explicit computation of the observables by means of the standard field theory formulation of gauge theories presents some technical difficulties, which are related to the gauge-fixing procedure and the definition of the fields propagator. For example, when $M = S^1 \times S^2$, the Feynman propagator for the A field does not exist because of the presence of a physical zero mode; in facts, among the field configurations, a globally defined 1-form A_0 exists such that $dA_0 = 0$ but A_0 is not the gauge transformed of something else. One can presumably overcome these technical difficulties, and one can imagine of computing the observables by means of perturbation theory. But, as a matter of facts, an explicit path-integral computation of the link observables (3.17) by means of the standard gauge theory perturbative methods has never been produced when the 3-manifold is not equal to $\mathbb{R}^3$.

For a nontrivial 3-manifold M, the expectation values $\langle W(L)\rangle\big|_M$ can be really computed —for certain manifolds— by using two methods: (i) a nonperturbative path-integral formalism based on the Deligne-Beilinson cohomology, (ii) the operator surgery method. In all the cases considered so far, these two methods give exactly the same answer.

Nonperturbative path-integral computation.

Let us consider a class of torsion-free manifolds of the type $S^1 \times \Sigma$, where Σ denotes the 2-sphere S^2 or a closed Riemann surface of genus $g \geq 1$. In this case, the first homology group $H_1(M)$ is not trivial and is given by the product of free abelian group factors; standard perturbation theory cannot be used since the Feynman propagator for the A field does not exist in $S^1 \times \Sigma$. But one can use the nonperturbative method developed in [49], in which the introduction of a gauge fixing and of the Feynman propagator is not necessary. The structure of the bundle $H^1_D(M)$, which is determined by the sequence (3.18), and of the resulting path-integral have been described in [49]. One finds:

1. when L is not homologically trivial (mod $2k$) in $S^1 \times \Sigma$,
$$\langle W(L)\rangle\Big|_{S^1\times\Sigma} = 0 \; ; \tag{3.23}$$
2. when L is homologically trivial (mod $2k$) in $S^1 \times \Sigma$,
$$\langle W(L)\rangle\Big|_{S^1\times\Sigma} = \exp\Big\{-(2i\pi/4k)\sum_{ij} q_i \mathbb{L}_{ij} q_j\Big\} \; , \tag{3.24}$$

which formally coincides with expression (3.22). Note that, when L is homologically trivial (mod $2k$), expression (3.24) is well defined [49].

By using nonperturbative path-integral arguments, the results shown in equations (3.23) and (3.24) have been generalized by Thuillier [57] to the case in which the 3-manifold is $M = RP^3$. This example is interesting because $H_1(RP^3)$ is not freely generated (in fact, $H_1(RP^3) = \mathbb{Z}_2$) and then RP^3 has nontrivial torsion.

Operator surgery method.

By means of the quantum groups modular algebra, one can construct link invariants of ambient isotopy; in order to compute these invariants in a nontrivial manifold M, Reshetikhin and Turaev have introduced appropriate surgery rules [58]. These rules —that have been also developed by Kohno [59], by Lickorish [60] and by Morton and Strickland [61] in the mathematical setting— have been adapted to the physical context in [44, 54, 55, 53]. We shall now recall the main features of the operator surgery method, which can be used to compute the abelian link invariants in a generic manifold M.

Every closed orientable connected 3-manifold M can be obtained by Dehn surgery on S^3 and admits a surgery presentation [62] which is described by a framed surgery link $\mathcal{L} \subset S^3$. A so-called surgery coefficient a_i is associated with each component $\mathcal{L}_i$ of $\mathcal{L}$; when a_i is an integer, we will put $a_i = \ell k(\mathcal{L}_i, \mathcal{L}_{i\mathrm{f}})$. For each manifold M, the corresponding surgery link $\mathcal{L}$ is not unique; all the possible surgery links which describe —up to orientation-preserving homeomorphisms— the same manifold are related by Kirby moves [62]. Any oriented coloured framed link $L \subset M$ can be described by a link $L' = L \cup \mathcal{L}$ in S^3 in which:

- the surgery link $\mathcal{L}$ describes the surgery instruction corresponding to a presentation of M in terms of Dehn surgery on S^3;
- the link L, which belongs to the complement of $\mathcal{L}$ in S^3, describes how L is actually placed in M.

According to the rules [53] of the operator surgery method, the expectation value of the Wilson line operator $W(L)$ in M can be written as a ratio

$$\langle W(L) \rangle \Big|_M = \langle W(L)\, W(\mathcal{L}) \rangle \Big|_{S^3} \Big/ \langle W(\mathcal{L}) \rangle \Big|_{S^3} , \tag{3.25}$$

where to each component of the surgery link $\mathcal{L}$ is associated a particular colour state ψ_0. Expression (3.22) implies that, for fixed integer k, the colour space of each link component coincides with space of residue classes of integers mod $2k$ (see also section 2.3). Thus the colour space has a canonical ring $\mathcal{R}$ structure; let χ_j denote the residue class associated with the integer j. Then, when the gauge group is $U(1)$, the colour state $\psi_0 \in \mathcal{R}$ is given by

$$\psi_0 = \sum_{j=0}^{2k-1} \chi_j \, . \tag{3.26}$$

This simply means that, in the computation of the observables (3.25), one must sum over the values $q = 0, 1, 2, ..., 2k - 1$ of the colours which are associated with the components of the surgery link (see for instance equation (3.44)). The proof that the surgery rules (3.25) and (3.26) are well defined and consistent —when the denominator of expression (3.25) is not vanishing— is nontrivial and essentially consists in proving that expression (3.25) is invariant under Kirby moves [63, 58, 53].

REMARK 2.1

The existence of surgery rules for the computation of the observables in quantum field theory is quite remarkable. Let us summarize the reasons for the existence of surgery rules in the Chern-Simons theory. Any 3-manifold M can be obtained [62] by removing and gluing back —after the introduction of appropriate homeomorphisms on their boundaries— solid tori embedded in S^3. The crucial point now is that the set of all possible surgeries is generated by two elementary operations which in facts correspond to twist homeomorphisms [62]. The action of these two twist homeomorphism generators on the observables can be found by analysing expression (3.22). This means that the solution of the Chern-Simons field theory in S^3 determines [53] the representation of the surgery on the set of observables. As a result, one can then connect the values of the observables in any nontrivial 3-manifold M with the values of the observables in S^3. For this reason, the solution of the topological Chern-Simons field theory in S^3 actually fixes the solution of the same theory in any closed oriented 3-manifold M.

3.3.3 Main properties

We conclude this section by recalling a few properties of the observables that will be useful for the following discussion. Since the linking numbers take integer values, expression (3.22) is invariant under the replacement $q_j \to q_j + 2k$, where q_j denotes the colour of a generic link component. Thus, for fixed k, the colour space of each link component can be identified with $\mathbb{Z}_{2k}$, which coincides with the space of the residue classes of integers mod $2k$. This property also holds [49] for the observables in a generic manifold M.

At the classical level, one link component C with colour $q > 1$ can be interpreted as the q-fold covering of C. At the quantum level, one needs to specify this correspondence a bit more precisely because of possible ambiguities in the computation of the expectation values of the composite Wilson lines operators. As we have already mentioned, all these ambiguities are removed by means of the framing procedure.

Satellites.

A general discussion of the satellite properties of the observables of the Chern-Simons theory can be found in Ref.[44, 58, 60, 61, 53]. Here we shall concentrate on the aspects which are relevant for the following exposition.

Let $C_{\rm f}$ be the framing of the oriented link component $C \subset M$ which has colour q with $|q| > 1$. One can imagine that C and $C_{\rm f}$ define the boundary of a band $\mathcal{B} \subset M$; then, one can [53, 49] simply replace C with $|q|$ parallel components $\{\widetilde{C}_1, ..., \widetilde{C}_{|q|}\}$ on $\mathcal{B}$ where each component has colour $q' = 1$ (in order to agree with the sign of q, one possibly needs to modify the orientations of the link components). The framings $\{\widetilde{C}_{1\rm f}, ..., \widetilde{C}_{|q|\rm f}\}$ of the components $\{\widetilde{C}_1, ..., \widetilde{C}_{|q|}\}$ also belong to the band $\mathcal{B}$. One can easily verify that the observables (3.22) are invariant under this substitution. To sum up, as far as the abelian link invariants are concerned, each link component C with colour $|q| > 1$ can always be interpreted as (and can be substituted with) the union of $|q|$ parallel copies of C with unitary colours.

DEFINITION 2.2 For any coloured, oriented and framed link $L \subset M$, one can introduce a new link $\widetilde{L} \subset M$ which is a satellite of L and which is obtained from L by replacing each link component of colour q with $|q|$ parallel copies of the same component, each copy with unitary colour. We call $\widetilde{L}$ the *simplicial satellite* of L.

The observables associated with any link L and the observables associated with its simplicial satellite $\widetilde{L}$ are totally equivalent. In other words, the observables of the abelian Chern-Simons theory in a generic manifold M satisfy [49] the relation

$$\langle W(L) \rangle \Big|_M = \langle W(\widetilde{L}) \rangle \Big|_M \; , \; \forall L \subset M \; . \tag{3.27}$$

The introduction of the simplicial satellites is useful because, in this way, we can possibly do without the concept of colour space, which has not a topological nature, and we can interpret the abelian link invariants entirely in terms of homology groups. This issue will be discussed in the next section.

3.4 Homology and link complements

In this section we show that, for any link $L \subset S^3$ with simplicial satellite $\widetilde{L}$, the abelian link invariant $\langle W(L)\rangle\big|_{S^3}$ is completely determined by the homology group $H_1(S^3 - \widetilde{L})$ of the complement of the link $\widetilde{L}$ in S^3. We also prove that

- the sets of the abelian Chern-Simons observables in each homology 3-sphere and in S^3 coincide;
- if the simplicial satellite of a link in a generic 3-manifold is homologically trivial, the associated observable coincides with an observable in S^3.

3.4.1 Link complements

Let us firstly recall that the homology group $H_1(X)$ of a manifold X can be interpreted as the abelianization of the fundamental group $\pi_1(X)$ because, given a presentation of $\pi_1(X)$ in terms of generators $\{\gamma_1, \gamma_2, ...\}$ and a set of relations between them, by adding the new constraints $[\gamma_a, \gamma_b] = 0$ for all a and b, one obtains a presentation of $H_1(X)$. Thus, let C_1 be an oriented knot in S^3; the homology group of its complement $X = S^3 - C_1$ is freely generated, $H_1(S^3 - C_1) = \mathbb{Z}$, and one can represent the generator g_1 by means of a small oriented circle C_{g_1} in S^3 linked with C_1 so that $\ell k(C_{g_1}, C_1) = 1$. Consider now a second oriented knot $C_2 \subset S^3 - C_1$, the class $[C_2]$ of C_2 in $H_1(S^3 - C_1)$ is just determined by the linking number of C_1 and C_2. Indeed, by using additive notations, one has

$$[C_2] = n\, g_1 \quad \Longleftrightarrow \quad \ell k(C_2, C_1) = n \; . \tag{3.28}$$

Moreover, if $C_{2\mathrm{f}}$ is a framing for C_2, one finds

$$H_1(S^3 - C_1) \ni [C_{2\mathrm{f}}] = [C_2] \quad , \quad \ell k(C_2, C_1) = \ell k(C_{2\mathrm{f}}, C_1) \; . \tag{3.29}$$

Let us now consider a framed, oriented and coloured link L with simplicial satellite $\widetilde{L} \subset S^3$, the associated abelian link invariant is given by

$$\langle W(L)\rangle\big|_{S^3} = \langle W(\widetilde{L})\rangle\big|_{S^3} = \exp\Big\{-(2i\pi/4k)\sum_{ij} \widetilde{\mathbb{L}}_{ij}\Big\} \, . \tag{3.30}$$

$\widetilde{\mathbb{L}}_{ij}$ denotes the linking matrix of $\widetilde{L}$ which can be written as

$$\widetilde{\mathbb{L}}_{ij} = \ell k(\widetilde{C}_{i\mathrm{f}}, \widetilde{C}_j) \; , \tag{3.31}$$

where $\widetilde{C}_j$ represents the j-th component of $\widetilde{L}$ and $\widetilde{C}_{i\mathrm{f}}$ is the framing of the component $\widetilde{C}_i \subset \widetilde{L}$. If g_j denotes the j-th generator of $H_1(S^3 - \widetilde{L})$ which is associated with the component $\widetilde{C}_j$, the class $[\widetilde{C}_{i\mathrm{f}}] \in H_1(S^3 - \widetilde{L})$ of the component $\widetilde{C}_{i\mathrm{f}}$ can be written as

$$[\widetilde{C}_{i\mathrm{f}}] = \sum_j \ell k(\widetilde{C}_{i\mathrm{f}}, \widetilde{C}_j)\, g_j \; . \tag{3.32}$$

So, the class $[\widetilde{L}_\mathrm{f}] \in H_1(S^3 - \widetilde{L})$ of the link $\widetilde{L}_\mathrm{f}$, which is the union of the framings

$$\widetilde{L}_\mathrm{f} = \bigcup_i \widetilde{C}_{i\mathrm{f}} \; , \tag{3.33}$$

is just

$$[\widetilde{L}_\mathrm{f}] = \sum_{i,j} \ell k(\widetilde{C}_{i\mathrm{f}}, \widetilde{C}_j)\, g_j = \sum_{i,j} \widetilde{\mathbb{L}}_{ij}\, g_j \; . \tag{3.34}$$

By comparing equations (3.30) and (3.34) one finds that the value of the abelian link invariant $\langle W(L)\rangle\big|_{S^3} = W(\widetilde{L})\rangle\big|_{S^3}$ is completely determined by the homology class $[\widetilde{L}_\mathrm{f}]$ of $\widetilde{L}_\mathrm{f}$ in $H_1(S^3 - \widetilde{L})$.

The abelian link invariant (3.30) also admits the following interpretation. Let $S_{\widetilde{L}}$ be a Seifert surface associated with the link $\widetilde{L} \subset S^3$; $S_{\widetilde{L}}$ is connected oriented (bicollared) with boundary $\partial S_{\widetilde{L}} = \widetilde{L}$. Let us denote by $\widetilde{L}_\mathrm{f} \cap S_{\widetilde{L}}$ the sum —by taking into account the signs— of the intersections of the link $\widetilde{L}_\mathrm{f}$ with the surface $S_{\widetilde{L}}$. Then, as a consequence of a possible definition of the linking number [62], one has

$$\langle W(L)\rangle\big|_{S^3} = \langle W(\widetilde{L})\rangle\big|_{S^3} = \exp\Big\{-(2i\pi/4k)\, \widetilde{L}_\mathrm{f} \cap S_{\widetilde{L}}\Big\} \, . \tag{3.35}$$

3.4.2 Sum of knots and cyclic covering

We now describe another possible interpretation of the abelian link invariants which makes use of the coverings of the complement of the links. Let us first introduce the concept of sum of knots.

DEFINITION 3.1. Let C_1 and C_2 be two oriented and framed components of a link L, and let both components C_1 and C_2 have the same colour q. By joining C_1 and C_2 in the way shown in Figure 1, one obtains the knot $C_1 \# C_2$, that we call the *sum of* C_1 *and* C_2. The framing $(C_1\#C_2)_\mathrm{f}$ of $C_1\#C_2$ is defined to be the sum of the framings $C_{1\mathrm{f}}\#C_{2\mathrm{f}}$ so that

$$\ell k((C_1\#C_2)_\mathrm{f}, C_1\#C_2) = \ell k(C_{1\mathrm{f}}, C_1) + \ell k(C_{2\mathrm{f}}, C_2) + 2\,\ell k(C_1, C_2) \; . \tag{3.36}$$

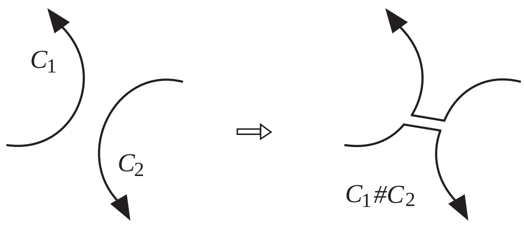

Figure 3.1: Sum of knots C_1 and C_2.

REMARK 3.2. Note that, when C_1 and C_2 belong to disjoint balls, the sum $C_1\#C_2$ coincides with the connected sum [62] of C_1 and C_2; in general, C_1 and C_2 may be linked and tied together. Note also that the linking number of $C_1\#C_2$ with a generic component C_j of the link L, with $j \geq 3$, is just the sum of the linking numbers

$$\ell k(C_1\#C_2, Cj) = \ell k(C_1, C_j) + \ell k(C_2, C_j) \ . \tag{3.37}$$

Now, the value of each observable (3.22) is invariant under the replacement of C_1 and C_2 by their sum $C_1\#C_2$. Indeed, as a consequence of the substitution of C_1 and C_2 with the sum $C_1\#C_2$, the linking matrix gets modified; instead of the first two rows and the first two columns of $\mathbb{L}_{ij}$ one has a new single row and a new single columm. But the relations (3.36) and (3.37) imply that the sum $\sum_{ij} q_i \mathbb{L}_{ij} q_j$ remains unchanged.

DEFINITION 3.3. For each coloured, oriented and framed link L in S^3, consider its simplicial satellite $\widetilde{L}$. All the components of $\widetilde{L}$ have the same (unitary) colour; therefore, one can recursively take the sum of the components of $\widetilde{L}$ so that, in the end, one obtains a single knot $L^\#$ that we call an *equivalent knot* of L.
By construction

$$\langle W(L) \rangle \Big|_{S^3} = \langle W(L^\#) \rangle \Big|_{S^3} \ , \ \ \forall L \subset S^3 \ . \tag{3.38}$$

Consider now an equivalent knot $L^\#$ of the link L and let $L^\#_{\rm f}$ be the framing of $L^\#$. From equation (3.38) it follows

$$\langle W(L) \rangle \big|_{S^3} = \langle W(L^\#) \rangle \big|_{S^3} = \exp\Big\{ -(2i\pi/4k)\, \ell k(L^\#_{\rm f}, L^\#) \Big\} \, . \tag{3.39}$$

This equation shows that the expectation value $\langle W(L) \rangle \big|_{S^3}$ is fixed by the homology group $H_1(S^3 - L^\#)$.

Finally, let $S_{L^\#}$ be a Seifert surface associated with $L^\# \subset S^3$. Let $\pi : Y_\infty \to Y$ be the infinite cyclic cover of $Y = S^3 - L^\#$ and let τ be the generator of covering translations which generates $\mathrm{Aut}(Y_\infty)$. Y_∞ can be obtained [62] by gluing —in a orientation preserving way— infinite copies of $S^3 - S_{L^\#}$ along (the images of) the surface $S_{L^\#}$. Consider $L_{\rm f}^\#$ as a loop in Y based at, for example, the point $y_0 \in L_{\rm f}^\#$. Let the path $(L_{\rm f}^\#)_\infty$ in Y_∞ be the lifting of $L_{\rm f}^\#$ based at a chosen point $(y_0)_\infty \in \pi^{-1}(y_0)$ and let $(y_1)_\infty \in \pi^{-1}(y_0)$ be the terminal point of the path $(L_{\rm f}^\#)_\infty$. There is a unique integer n such that $\tau^n(y_0)_\infty = (y_1)_\infty$; this integer precisely determines the value of the observable

$$\langle W(L)\rangle\big|_{S^3} = \langle W(L^\#)\rangle\big|_{S^3} = \exp\Big\{-(2i\pi/4k)\, n\Big\} \tag{3.40}$$

because, in agreement with equation (3.35), in going along $L_{\rm f}^\#$, n simply counts (by taking into account the signs) how many times one runs across the Seifert surface $S_{L^\#}$. Expression (3.40) is periodic in n with period $4k$; so, for fixed integer k, instead of the infinite cyclic cover Y_∞, one can actually consider the $4k$-fold cyclic cover of $Y = S^3 - L^\#$.

3.4.3 Homology spheres

In order to study the properties of the observables in homology spheres, we need to recall the meaning of the surgery instruction which is described by a framed surgery link $\mathcal{L} \subset S^3$. Let $\{\mathcal{L}_i\}$ (with $i = 1, 2, ..., N_\mathcal{L}$) be the link components of $\mathcal{L}$ with framings $\{\mathcal{L}_{i\rm f}\}$. The 3-manifold $M_\mathcal{L}$ which corresponds to $\mathcal{L}$ can be obtained by means of the following operations; for each link component $\mathcal{L}_i$,

- remove from S^3 the interior $\mathring{V}_i$ of a tubular neighbourhood V_i of the component $\mathcal{L}_i$;
- sew the solid torus V_i on $S^3 - \mathring{V}_i$ by means of the boundaries identification given by a homeomorphism $h_i : \partial V_i \to \partial(S^3 - \mathring{V}_i)$ which sends a meridian μ_i of V_i into the framing $\mathcal{L}_{i\rm f}$ of $\mathcal{L}_i$, i.e. $\mathcal{L}_{i\rm f} = h_i(\mu_i) \in \partial(S^3 - \mathring{V}_i)$.

One example of surgery is depicted in Figure 2; in this case, the surgery link coincides with the trefoil knot with surgery coefficient 2.

Let g_i be the generator of $H_1(S^3 - \mathcal{L})$ which is associated with the link component $\mathcal{L}_i$, with $i = 1, 2, ..., N_\mathcal{L}$, where an orientation has been introduced for each $\mathcal{L}_i$. Since the meridians $\{\mu_i\}$ of the tubular neighbourhoods $\{V_i\}$ are homologically trivial, their images $\mathcal{L}_{i\rm f} = h_i(\mu_i)$ also must be homologically trivial. So, the homology group $H_1(M_\mathcal{L})$ of the manifold $M_\mathcal{L}$ admits

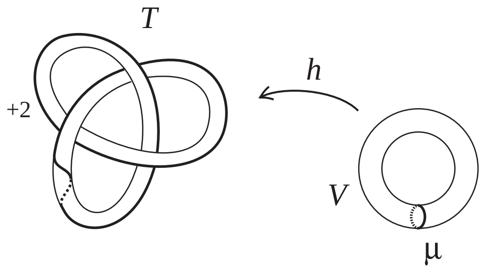

Figure 3.2: One example of surgery over the trefoil T with surgery coefficient $+2$; according to the homeomorphism h, the image of the meridian μ of the solid torus V is a framing of T with linking number $+2$ with T.

the following presentation —with $N_{\mathcal{L}}$ generators and at most $N_{\mathcal{L}}$ nontrivial relations—

$$H_1(M_{\mathcal{L}}) = \langle\, g_1, g_2, \cdots, g_{N_{\mathcal{L}}} \,|\, [\mathcal{L}_{1\mathrm{f}}] = 0, [\mathcal{L}_{2\mathrm{f}}] = 0, \cdots \rangle \;, \tag{3.41}$$

with $[\mathcal{L}_{i\mathrm{f}}] \in H_1(S^3 - \mathcal{L})$ for $i = 1, 2, ..., N_{\mathcal{L}}$. If $M_{\mathcal{L}}$ is a homology sphere, $H_1(M_{\mathcal{L}})$ must be trivial. In this case, the set of relations

$$[\mathcal{L}_{i\mathrm{f}}] \equiv \sum_j \ell k(\mathcal{L}_{i\mathrm{f}}, \mathcal{L}_j)\, g_j = 0 \quad , \qquad \text{for } i = 1, 2, ..., N_{\mathcal{L}} \tag{3.42}$$

must only admit the trivial solution $g_1 = g_2 = \cdots = 0$. This means that, when —by means of Kirby moves— the linking matrix of the surgery link is reduced in diagonal form, each diagonal matrix element must coincide with $+1$ or -1. In fact, the following theorem has been proved [64].

THEOREM 3.4. *Each homology 3-sphere admits a surgery presentation in S^3 described by a surgery link $\mathcal{L}$ which is algebraically split (i.e. $\ell k(\mathcal{L}_i, \mathcal{L}_j) = 0\ \forall\, i \neq j$) and has surgery coefficients equal to ± 1 (i.e. $\ell k(\mathcal{L}_{i\mathrm{f}}, \mathcal{L}_i) = \pm 1\ \forall\, i$).*

We can now demonstrate the following result.

THEOREM 3.5. *The sets of the abelian Chern-Simons observables in S^3 and in any homology 3-sphere M_0 ($H_1(M_0) = 0$) coincide,*

$$\left\{ \langle W(L) \rangle \Big|_{M_0} \right\} = \left\{ \langle W(L) \rangle \Big|_{S^3} \right\} . \tag{3.43}$$

PROOF. Let M_0 be a homology sphere and let $\mathcal{L} \subset S^3$ be a surgery link, which corresponds to a surgery presentation of M_0 in S^3, such that the properties specified by Theorem 3.4 are satisfied (that is, $\mathcal{L}$ is algebraically split with surgery coefficients ± 1). Any link L in M_0 can be described by a link, that we shall also denote by L, in the complement of $\mathcal{L}$ in S^3. In order to compute the observable $\langle W(L)\rangle\big|_{M_0}$ we shall use the surgery method described in equation (3.25). The denominator of expression (3.25) contains the expectation value

$$\begin{aligned}\langle W(\mathcal{L})\rangle\Big|_{S^3} &= \textstyle\sum_{q_1,q_2,\ldots} \exp\Big\{-(2i\pi/4k)\sum_{ij} q_i\, \ell k(\mathcal{L}_{\rm if},\mathcal{L}_j)\, q_j\Big\} = \\ &= \textstyle\prod_i \sum_{q_i} \exp\Big\{-(2i\pi/4k) q_i^2\, \ell k(\mathcal{L}_{\rm if},\mathcal{L}_i)\Big\}\ .\end{aligned} \qquad (3.44)$$

Let us consider each term of the product entering equation (3.44); since $\ell k(\mathcal{L}_{\rm if},\mathcal{L}_i) = \pm 1$ one finds [65]

$$\sum_{q=0}^{2k-1} \exp\Big\{\pm(2i\pi/4k)q^2\Big\} = e^{\pm i\pi/4}\sqrt{2k}\ , \qquad (3.45)$$

and then $\langle W(\mathcal{L})\rangle\big|_{S^3} \neq 0$. This means that equation (3.25) is well defined; let us now consider the numerator of equation (3.25).

Let us denote by L_α, with $\alpha = 1, 2, ..., N_L$, the α-th component of the link L with colour q_α, and let

$$t_i = \sum_\alpha q_\alpha\, \ell k(\mathcal{L}_i, L_\alpha)\ . \qquad (3.46)$$

Then, in the computation of the numerator $\langle W(L)W(\mathcal{L})\rangle\big|_{S^3}$ of the ratio (3.25), the contribution of the generic component $\mathcal{L}_i$ of the surgery link $\mathcal{L}$ is given by the multiplicative factor

$$\sum_{q_i=0}^{2k-1} \exp\Big\{-(2i\pi/4k)\,\big[(\pm q_i^2) + 2q_i t_i\big]\Big\} = e^{(\mp)i\pi/4}\sqrt{2k}\ \exp\Big\{-(2i\pi/4k)(\mp t_i^2)\Big\}\ . \qquad (3.47)$$

In the computation of ratio (3.25), the term $e^{(\mp)i\pi/4}\sqrt{2k}$ cancels with the same factor appearing in the denominator, see equation (3.45). Whereas the remaining term $\exp\{-(2i\pi/4k)(\mp t_i^2)\}$ corresponds to the effect of a (∓ 1) twist homeomorphism of the link components of L which are linked with $\mathcal{L}_i$.

So, in the computation of $\langle W(L)\rangle\big|_{M_0}$, the global effect of the surgery link $\mathcal{L}$ is just to introduce of certain number of twist homeomorphisms on the link L whose expectation value has eventually to be computed in S^3. This means that, for each link $L \subset M_0$ one finds a suitable link $L' \subset S^3$ such that $\langle W(L)\rangle\big|_{M_0} = \langle W(L')\rangle\big|_{S^3}$. Consequently, the sets of expectation values $\Big\{\langle W(L)\rangle\big|_{M_0}\Big\}$ and $\big\{\langle W(L)\rangle\big|_{S^3}\big\}$ coincide; this concludes the proof. □

REMARK 3.6. We would like to present now a different proof of Theorem 3.5 which is not based on algebraic manipulations. The new proof makes use of the properties of the Kirby moves and is entirely based on the fact that the abelian link invariants only depend on the linking numbers between the link components. The starting point is that a function of the abelian link invariants, which provides a realization of the surgery rules, exists (equations (3.25) and (3.26)). Let us consider a surgery presentation of the homology sphere M_0 in S^3 which is described by a surgery link $\mathcal{L}$; according to Theorem 3.4.3, one can assume that $\mathcal{L}$ is algebraically split and has surgery coefficients ± 1. Since all the linking numbers between the link components of $\mathcal{L}$ are vanishing, the components $\mathcal{L}_i$ can be untied so that one obtains the distant union of knots, each with surgery coefficient ± 1. By means of a finite number of overcrossing/undercrossing exchanges, each knot can be unknotted. Thus, for each surgery knot one can introduce [62] (by means of Kirby moves) a finite number of new elementary surgery components —which are given by unknots with surgery coefficients ± 1— which unknot the knot and have vanishing linking number with the knot. One example of this move is shown in Figure 3.

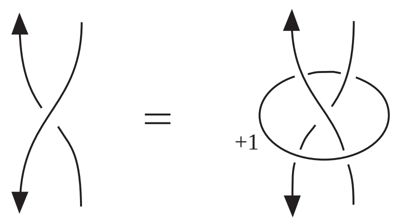

Figure 3.3: One example of Kirby move: by means of the introduction of the new surgery component, an undercrossing is replaced by an overcrossing.

Again, since all the linking numbers are vanishing, these new link components also can be untied completely. As a result, the entire set of surgery instructions is effectively described by the distant union of unknots with surgery coefficients ± 1. The action of this surgery on S^3 is trivial, it maps S^3 into S^3, because it simply introduces a set of disjoint elementary (± 1) twist homemorphisms which possibly act on the links $L \subset S^3$. This is precisely in agreement with the conclusions that have been obtained in the previous algebraic version of the proof. So, the set of the abelian Chern-Simons observables in a generic homology 3-sphere M_0 coincides with the set of observables in the 3-sphere S^3.

3.4.4 Homologically trivial links

In this section we shall consider homologically trivial links in a generic manifold M. Let us firstly introduce a general property of the expectation values.

PROPERTY 3.7. Suppose that the link L_U is the union of the link L and of the unknot U in a generic manifold M, $L_U = L \cup U \subset M$. If the unknot U belongs to a 3-ball which is disjoint from the link L and U has trivial framing (i.e. its framing $U_{\rm f}$ satisfies $\ell k(U, U_{\rm f}) = 0$), then, independently of the colour q associated with U, one finds

$$\langle W(L_U)\rangle\Big|_M = \langle W(U)W(L)\rangle\Big|_M = \langle W(L)\rangle\Big|_M \; . \tag{3.48}$$

PROOF. When $M = S^3$, equation (3.48) follows immediately from expression (3.22). In the case of a generic 3-manifold M, equation (3.48) is a consequence of definition (3.25) and of the ambient isotopy invariance of the observables. In facts, if U belongs to a 3-ball, U is ambient isotopic with an unknot which belongs to a 3-ball which is disjoint from the surgery link $\mathcal{L}$ entering equation (3.25) —and then this unknot is not linked with $\mathcal{L}$. Moreover, since L_U is the distant union of U and L, the unknot U is not linked with the components of the link L; finally, U has trivial framing and then equality (3.48) follows. □

Let us now consider the observable $\langle W(L)\rangle\big|_M$ which is associated with a link L in a generic 3-manifold M.

THEOREM 3.8. *Let M be a generic closed and oriented 3-manifold; if the simplicial satellite $\widetilde{L} \subset M$ of the link L is homologically trivial and the associated observable (3.25) is well defined, then there exists a link —that we denote by L'— in S^3 such that*

$$\langle W(L)\rangle\Big|_M = \langle W(L')\rangle\Big|_{S^3} \; . \tag{3.49}$$

PROOF. In agreement with the surgery recipe of equations (3.25) and (3.26), one has

$$\langle W(L)\rangle\Big|_M = \langle W(\widetilde{L})\rangle\Big|_M = \langle W(\widetilde{L})\, W(\mathcal{L})\rangle\Big|_{S^3} \Big/ \langle W(\mathcal{L})\rangle\Big|_{S^3} \, , \tag{3.50}$$

where $\widetilde{L}$ is the simplicial satellite of L and $\mathcal{L}$ is a surgery link —with components $\{\mathcal{L}_i\}$— which corresponds to M. We assume that expression (3.50) is well defined. Let us consider the presentation (3.41) of the group $H_1(M)$. If the link $\widetilde{L}$ is homologically trivial in M, the class $[\widetilde{L}]$ of $\widetilde{L}$ in $H_1(S^3 - \mathcal{L})$ can be written in the form

$$H_1(S^3 - \mathcal{L}) \ni [\widetilde{L}] = \sum_i n_i \, [\mathcal{L}_{i{\rm f}}] \; , \;\; \text{with } n_i \in \mathbb{Z} \; . \tag{3.51}$$

where $[\mathcal{L}_{i\text{f}}] \in H_1(S^3 - \mathcal{L})$. Indeed, each component $\mathcal{L}_{i\text{f}}$ is homologically trivial in $H_1(M)$ and, in the presentation (3.41), all the constraints are precisely generated by the equations $[\mathcal{L}_{i\text{f}}] = 0$ for $i = 1, 2, ..., N_{\mathcal{L}}$.
Let us now consider the new link $L' \subset M$ which, in the surgery presentation of M, is described by the link $L' \subset S^3 - \mathcal{L}$ given by

$$L' = \widetilde{L} \cup K_1 \cup K_2 \cup \cdots \cup K_m \; , \tag{3.52}$$

where each knot K_j, with $j = 1, 2, ..., m$, is an unknot with unitary colour and $m = \sum_j n_j$. More precisely, each of the first n_1 unknots of equation (3.52), $K_1, K_2, ..., K_{n_1}$, is ambient isotopic with $\mathcal{L}_{1\text{f}}$ with reversed orientation; each of the next n_2 unknots $K_{n_1+1}, K_{n_1+2}, ..., K_{n_1+n_2}$, is ambient isotopic with $\mathcal{L}_{2\text{f}}$ with reversed orientation and so on. If K_j is ambient isotopic with $\mathcal{L}_{i\text{f}}$ with reversed orientation, the framing $K_{j\text{f}}$ of K_j is chosen in such a way that $\ell k(K_j, K_{j\text{f}}) = \ell k(\mathcal{L}_i, \mathcal{L}_{i\text{f}})$. According to the surgery instructions described in section 3.3, each component $\mathcal{L}_{i\text{f}}$ is homeomorphic with a meridian of a solid torus and then $\mathcal{L}_{i\text{f}}$ is ambient isotopic with an unknot which belong to a 3-ball that is disjoint from all the remaining link components. Consequently, each knot K_j is ambient isotopic with an unknot which belongs to a 3-ball in M and, by construction, this unknot has trivial framing. So, in agreement with the Property 3.7, one has

$$\langle W(L) \rangle \Big|_M = \langle W(L') \rangle \Big|_M = \langle W(L') \, W(\mathcal{L}) \rangle \Big|_{S^3} \Big/ \langle W(\mathcal{L}) \rangle \Big|_{S^3} . \tag{3.53}$$

The class $[L']$ of L' in $S^3 - \mathcal{L}$ follows from the definition (3.52) and equation (3.51),

$$H_1(S^3 - \mathcal{L}) \ni [L'] = \sum_i n_i \, [\mathcal{L}_{i\text{f}}] + \sum_j [K_j] = 0 \; . \tag{3.54}$$

This means that L' —or its equivalent knot $L'^{\#}$— is not linked with each of the components of the surgery link $\mathcal{L}$. Consequently, in the computation of the ratio (3.53), the expectation value $\langle W(\mathcal{L}) \rangle \big|_{S^3}$ factorizes in the numerator and cancels out with the denominator, and finally one obtains

$$\langle W(L) \rangle \Big|_M = \langle W(L') \Big|_{S^3} . \tag{3.55}$$

To sum up, if the simplicial satellite $\widetilde{L} \subset M$ of the link L is homologically trivial, there exists a link $L' \subset S^3$ such that equation (3.55) is satisfied, and this concludes the proof. Finally, because of the colour periodicity property of observables, for fixed integer k Theorem 3.8 actually holds when $\widetilde{L}$ is homologically trivial mod $2k$. □

3.5 Three-manifold invariants

The surgery rules and the 3-manifold invariant of Reshetikhin and Turaev [58] for the gauge group $SU(2)$ can be generalized to the case in which the gauge group is $U(1)$. Let $M = M_{\mathcal{L}}$ be the 3-manifold which is obtained by means of the Dehn surgery which is described by the surgery link $\mathcal{L}$ in S^3; the 3-manifold invariant $I_k(M)$ for the abelian $U(1)$ gauge group takes the form

$$I_k(M) = I_k(M_{\mathcal{L}}) = (2k)^{-N_{\mathcal{L}}/2}\; e^{i\pi\sigma(\mathcal{L})/4} \langle W(\mathcal{L})\rangle\Big|_{S^3} \, , \tag{3.56}$$

where $N_{\mathcal{L}}$ denotes the number of components of $\mathcal{L}$ and $\sigma(\mathcal{L})$ represents the so-called signature of the linking matrix associated with $\mathcal{L}$, i.e. $\sigma(\mathcal{L}) = n_+ - n_-$ where $n_\pm$ is the number of positive/negative eigenvalues of the linking matrix which is defined by the framed link $\mathcal{L}$. Expression (3.56) is invariant under Kirby moves [58, 61, 53] and therefore is invariant under orientation preserving homeomorphisms of the 3-manifold M. Note that the orientation of $M = M_{\mathcal{L}}$ is induced by the orientation of S^3 on which the surgery acts. A modification of the orientation of M is equivalent to the replacement of $I_k(M)$ by its complex conjugate $\overline{I_k(M)}$.

REMARK 4.1. The invariance under Kirby moves of expression (3.56) can be used to understand the consistency of the surgery rules (3.25) for the observables. In facts, if one multiplies the numerator and the denominator of equation (3.25) by the same factor $(2k)^{-N_{\mathcal{L}}/2}\; e^{i\pi\sigma(\mathcal{L})/4}$, the numerator and the denominator are separately invariant under Kirby moves.

Let us recall that, according to the prescription (3.26), in the computation of the expectation value $\langle W(\mathcal{L})\rangle\big|_{S^3}$ one must take the sum over the values $q = 0, 1, ..., 2k-1$ of the colour which is associated with each component of the surgery link $\mathcal{L}$. This just corresponds to the standard Reshetikhin-Turaev prescription in the case of gauge group U(1). Actually, for fixed integer k, the Reshetikhin-Turaev invariant (3.56) admits the following natural generalization.

DEFINITION 4.2. As we have already mentioned, for fixed integer k the colour space is isomorphic with $\mathbb{Z}_{2k}$. For each subgroup $\mathbb{Z}_p$ of $\mathbb{Z}_{2k}$, one can introduce the 3-manifold invariant $I_{(p,k)}(M)$ defined by

$$I_{(p,k)}(M) = I_{(p,k)}(M_{\mathcal{L}}) = a^{-N_{\mathcal{L}}/2}\, e^{i\varphi\sigma(\mathcal{L})} \langle W(\mathcal{L})\rangle_{(p,k)}\Big|_{S^3} \, , \tag{3.57}$$

where the positive real number a and the phase factor $e^{i\varphi}$ are determined by

$$\sum_{b=0}^{p-1} \exp\Big\{ -(i\pi\, d/p)\, b^2 \Big\} = a\, e^{-i\varphi} \, , \tag{3.58}$$

in which $d = 2k/p$. $\langle W(\mathcal{L})\rangle_{(p,k)}\big|_{S^3}$ denotes the sum of the expectation values when the colour of each link component takes the values $q = 0, d, 2d, ..., (p-1)d$. One has $I_{(2k,k)}(M_{\mathcal{L}}) = I_k(M_{\mathcal{L}})$.
The proof that $I_{(p,k)}(M_{\mathcal{L}})$ is invariant under Kirby moves is based precisely on the same steps that enter the corresponding proof for $I_k(M_{\mathcal{L}})$.

A field theory interpretation of the Reshetikhin-Turaev invariant (3.56) —as a ratio of Chern-Simons partition functions— has been proposed in [53] and detailed discussions on the properties of the invariant (3.56) can be found in [66, 67, 68].

PROPERTY 4.3. *If the manifold M_0 is a homology 3-sphere, then $I_k(M_0) = 1$.*

PROOF. By Theorem 3.4, M_0 admits a surgery presentation in S^3 in which the surgery link $\mathcal{L}$ is algebraically split with surgery coefficients ± 1. Since the link components of $\mathcal{L}$ are not linked, in agreement with equation (3.44) the expectation value $\langle W(\mathcal{L})\rangle\big|_{S^3}$ is just the product of terms shown in equation (3.45). These terms cancel with the normalization factor $(2k)^{-N_{\mathcal{L}}/2}\, e^{i\pi\sigma(\mathcal{L})/4}$ which is present in the definition of $I_k(M_0)$, and then $I_k(M_0) = 1$. □

Let us consider the lens spaces $L_{p/r}$ where the integers p and r are coprime and satisfy $0 < r < p$. The fundamental group of $L_{p/r}$ is $\mathbb{Z}_p$ and one also has $H_1(L_{p/r}) \simeq \mathbb{Z}_p$. When $p \neq p'$, the lens spaces $L_{p/r}$ and $L_{p'/r'}$ are not homeomorphic. The manifolds $L_{p/r}$ and $L_{p/r'}$ are homeomorphic iff $\pm r' \equiv r^{\pm 1}$ (mod p). The manifold L_p admit a surgery presentation given by the unknot with surgery coefficient equal to the integer p. Special cases are $L_0 \simeq S^2 \times S^1$, $L_1 \simeq S^3$; equation (3.56) gives

$$I_k(S^3) = 1 \quad , \quad I_k(S^2 \times S^1) = \sqrt{2k}\ . \tag{3.59}$$

By using the following reciprocity formula [69]

$$\sum_{n=0}^{|c|-1} e^{-i\frac{\pi}{c}(an^2+bn)} = \sqrt{|c/a|}\, e^{-i\frac{\pi}{4ac}(|ac|-b^2)} \sum_{n=0}^{|a|-1} e^{i\frac{\pi}{a}(cn^2+bn)}\ , \tag{3.60}$$

which is valid for integers a, b and c such that $ac \neq 0$ and $ac + b =$ even, one gets (for integer $p > 1$)

$$I_k(L_p) = \frac{1}{\sqrt{p}} \sum_{n=0}^{p-1} e^{2\pi ik\,(n^2/p)}\ . \tag{3.61}$$

Let us compare expression (3.61) with the functional integral interpretation [53] of the Reshetikhin-Turaev invariant

$$I_k(L_p) = \frac{\int_{L_p} DA\, e^{2\pi ikS}}{\int_{S^3} DA\, e^{2\pi ikS}}\ . \tag{3.62}$$

In agreement with equation (3.18), the fields configuration space $H^1_D(L_p)$ has the structure of a bundle over $H^2(L_p) = \mathbb{Z}_p$ with fibre $\Omega^1(L_p)/\Omega^1_{\mathbb{Z}}(L_p)$. Let the group $H^2(L_p)$ be generated by the element h, with $h^p = 1$. The path-integral (3.62) over $H^1_D(L_p)$ is formally given by a sum of p terms; the n-th term corresponds to the path-integral over 1-forms modulo forms of integer periods in the n-th sector of $H^1_D(L_p)$ which is associated to the element h^n of the second cohomology group of L_p. The result (3.61) suggests the possibility that the path-integral in the n-th sector of $H^1_D(L_p)$ is saturated by a single value $S\Big|_n$ of the Chern-Simons action, with $S\Big|_n = n^2/p$ modulo integers.

The manifold $\Sigma_g \times S^1$, where Σ_g denotes a closed oriented surface of genus g, admits a surgery presentation that is described [70] by a surgery link which contains $2g+1$ components (with vanishing surgery coefficients) and has vanishing linking matrix. The first homology group is $H_1(\Sigma_g \times S^1) = \mathbb{Z}^{2g+1}$; one finds

$$I_k(\Sigma_g \times S^1) = (2k)^{(2g+1)/2} \; . \tag{3.63}$$

Since the abelian link invariants only depend on the homology of the complement of the (simplicial satellites of) links in S^3, one could suspect that the invariant $I_k(M)$ only depends on the homology group $H_1(M)$ of the closed oriented 3-manifold M. This guess is supported by Property 4.3 and by the result (3.63); moreover, it naturally fits the structure of the configuration space on which the functional integral is based. However, a few counter-examples demonstrate that this conjecture is false. In the non-abelian case, this guess is not correct; in facts, explicit counter-examples for the gauge groups $SU(2)$ and $SU(3)$ can be found in Ref.[71].

The lens space $L_{5/1}$ admits a surgery presentation in S^3 which is described by the unknot with surgery coefficient 5; whereas a surgery link corresponding to $L_{5/2}$ is the Hopf link [62] in S^3 with surgery coefficients 2 and 3. From equation(3.56) we obtain

$$I_2(L_{5/1}) = -1 \qquad , \qquad I_2(L_{5/2}) = 1 \; . \tag{3.64}$$

The manifold $L_{9/1}$ can be described by the unknot in S^3 with surgery coefficient 9 and $L_{9/2}$ corresponds to the Hopf link with surgery coefficients 5 and 2. We get

$$I_3(L_{9/1}) = i\sqrt{3} \qquad , \qquad I_3(L_{9/2}) = -i\sqrt{3} \; . \tag{3.65}$$

Homotopy type

The manifolds $L_{9/1}$ and $L_{9/2}$ are of the same homotopy type [62]. Therefore, equation (3.65) also shows that the Reshetikhin-Turaev invariant (3.56) is not a function of the homotopy type of the manifold M only. In facts, Murakami, Ohtsuki and Okada have shown [66] that expression (3.56) is invariant under orientation-preserving homotopies [72]. Since under a modification of the orientation of the manifold M one gets $I_k(M) \to \overline{I_k(M)}$, the result (3.65) is in agreement with Murakami, Ohtsuki and Okada statement.

Let us consider the lens spaces with the orientation induced by the surgery presentation; $L_{p/r}$ and $L_{p/r'}$ are of the same homotopy type iff $\pm rr' \equiv m^2 \pmod p$ for some integer m. Hansen, Slingerland and Turner have shown [68] that, when $rr' \equiv -m^2 \pmod p$, one finds $I_k(L_{p/r}) = \overline{I_k(L_{p/r'})}$; one example of this kind is shown in equation (3.65). Whereas, when the product rr' is equivalent to a quadratic residue, $rr' \equiv m^2 \pmod p$, one has $I_k(L_{p/r}) = I_k(L_{p/r'})$, for istance

$$I_3(L_{7/1}) = -i \qquad , \qquad I_3(L_{7/2}) = -i \, . \tag{3.66}$$

The equivalence relation under orientation-preserving homotopy extends to the manifolds which are connected sum of equivalent spaces [72]. However, in the presence of orientation-reversing homotopy, this equivalence relation in general does not survive the connected sum. For instance, the connected sums $L_{9/1}\#L_{7/1}$ and $L_{9/2}\#L_{7/2}$ have different Reshetikhin-Turaev invariants which are not related by complex conjugation

$$I_3(L_{9/1}\#L_{7/1}) = \sqrt{3} \qquad , \qquad I_3(L_{9/2}\#L_{7/2}) = -\sqrt{3} \, . \tag{3.67}$$

Chapter 4

Thermodynamics of NACS Particles

4.1 Introduction

Unlike ordinary three-dimensional systems, quantum two-dimensional systems of indistinguishable particles allow for generalized braiding statistics. A celebrated generalization of the usual bosonic and fermionic quantum statistics is provided in two dimensions by Abelian anyons, for which a phase factor multiplying the scalar wavefunction is associated to elementary braiding operations [10, 73, 74]

$$\psi(z_1, .., z_i, \dots, z_j, .., z_n) = e^{i\pi\alpha}\, \psi(z_1, .., z_j, \dots, z_i, .., z_n) \ . \tag{4.1}$$

Anyons, first studied in [14, 75, 76], were later associated to the physics of the fractional quantum Hall effect [10]. Abelian anyon statistics of the simplest QH states, at filling factors $\nu = 1/(2p+1)$ were derived from a microscopic theory [77]: since then, the study of the properties of Abelian anyons and the applications to the QHE have been in the following decades subject of an intense and continuing interest [7, 78, 79, 80, 81].

A further generalization of the bosonic and fermionic statistics is represented by non-Abelian anyons, described by a multi-component wavefunction $\psi_a(z_1, \dots, z_n)$ $(a = 1, 2, \dots, g)$ which undergoes a linear unitary transformation under the effect of braiding σ_i which exchanges the particles at the positions z_i and z_{i+1}

$$\psi_a \to [\rho(\sigma_i)]_{ab}\, \psi_b \ , \tag{4.2}$$

where $\rho(\sigma_i)$ are $g \times g$ dimensional unitary matrices which do not commute among themselves, $[\rho(\sigma_i)]_{ab}[\rho(\sigma_j)]_{bc} \neq [\rho(\sigma_j)]_{ab}[\rho(\sigma_i)]_{bc}$ [74].

Abelian and non-Abelian anyons respectively correspond to one-dimensional and higher-dimensional representations of the braid group: with respect to parastatistics, non-Abelian anyons represent the counterpart of the generalization represented by Abelian anyons with respect to ordinary Bose and Fermi statistics. Non-Abelian anyons naturally appear in the description of a variety of physical phenomena, ranging from the fractional QHE [82, 74] to the scattering of vortices in spontaneously broken gauge theories [83, 84, 85], the (2+1)-dimensional gravity [30, 86, 87] and the alternation and interchange of $e/4$ and $e/2$ period interference oscillations in QH heterostructures [88].

The non-Abelian anyons studied in this work are non-Abelian Chern-Simons (NACS) spinless particles. The NACS particles, which are pointlike sources mutually interacting via a topological non-Abelian Aharonov-Bohm effect [89], carry non-Abelian charges and non-Abelian magnetic fluxes, so that they acquire fractional spins and obey braid statistics as non-Abelian anyons. More specifically, our models are described by the Hamiltonian (4.15) which involves the isovector operators Q^a_α in a representation of isospin l, where $\alpha = 1, 2, \ldots, N$ refers to any of the N particles of the system. With respect to the index α which labels the particles, these operators commute one to the other. Correspondingly, the quantum dimension of our anyonic systems is an integer number, contrary to what happens, for instance, in the Fibonacci anyons used to implement topological quantum computation [74], whose quantum dimension is instead an irrational number. Futhermore the NACS systems studied in this Chapter are gapless in the thermodynamic limit, contrary to the Fibonacci anyons or alike which have a gap in the bulk.

The study of equilibrium properties of two-dimensional anyonic systems is in general a non trivial and highly interesting task: indeed, the anyonic statistics incorporate the effects of interaction in microscopic bosonic or fermionic systems (statistical transmutation) so that the determination of thermodynamical properties of non-interacting anyons is at least as much as difficult as similar computation in ordinary interacting gas. This is a reason for which the investigation of equilibrium properties of a free gas of anyons called for an huge amount of efforts and work [90], the other reason of course being that the thermodynamics of a system of free anyons is the starting point - paradigmatic for the simplicity of the model - for the understanding of the thermodynamics of more complicated interacting anyon gas.

The two-dimensional gas of free Abelian anyons whose wavefunction fulfills hard-core wavefunction boundary conditions has been studied by Arovas, Schrieffer, Wilczek, and Zee [91] in its low-density regime by taking its virial expansion. In particular, they found the exact expression for the second virial coefficient, that turns out to be periodic and non-analytic as a function of the statistical parameter. Results for higher virial coefficients of the free Abelian gas are also available in literature: different approaches have been used, including the semiclassical approximation [92] and Monte Carlo computations [93] (for more references see [73, 94]). Useful results can be found by perturbative expansions in powers of the statistical parameter α: exact expressions for the first three terms of the expansions in powers of α are available for each of the first six virial coefficients [95, 96]. The second virial coefficient is the only one presenting - in each of the Bose points - cusps in the statistical parameter α [73], i.e. none of the higher virial coefficients have terms at order α [97, 98]. Furthermore, a recursive algorithm permits to compute the term in α^2 of all the cluster and virial coefficients [97, 98, 99, 100].

The results for the virial coefficients of the free gas of Abelian anyons quoted in the previous paragraph are obtained considering a many-body anyonic wavefunction fulfilling hard-core boundary conditions, i.e. a wave function which vanishes in correspondence of coincident points in the configuration space of the set of anyons. The generalization obtained by removing such an hard-core constraint has been studied for Abelian anyons [101, 102, 103] and a family of anyon models can be associated to the different boundary conditions of the same Hamiltonian. These models are obtained within the frame of the quantum-mechanical method of the self-adjoint extensions of the Schrödinger anyonic Hamiltonian. In the following we will refer to anyons without the constraint of hard-core conditions as *"soft-core"* or *"colliding"* anyons. The mathematical arguments underlying the possibility of such a generalization were discussed in [101], and the second virial coefficient of soft-core Abelian anyons was studied in [102, 103]. The corresponding self-adjoint extensions for the non-Abelian anyonic theory have been thoroughly discussed [104, 105, 106]. We stress that it is not easy, in general, to extract the parameters of emerging effective (eventually free) anyonic models from the microscopic Hamiltonians, and then the introduction of soft-core conditions may provide useful parameters which have to be fixed via the comparison between the results of the anyonic models and the computations done in the underlying microscopic models.

For non-Abelian anyons, a study of the thermodynamical properties in the lowest Landau level of a strong magnetic field has been performed [107], showing that the virial coefficients are independent of the statistics. The theory of non-relativistic matter with non-Abelian Chern-Simons gauge interaction in $(2+1)$ dimensions was studied adopting a mean field approximation in the current-algebra formulation already applied to the Abelian anyons and finding a superfluid phase [108].

In comparison with the Abelian case, the thermodynamics of a system of free non-Abelian anyons appears to be much harder to study and all the available results are for hard-core boundary conditions [109, 110, 111, 112], with - at the best of our knowledge - no results (even for the second virial coefficient) for soft-core non-Abelian anyons.

The reason of this gap is at least twofold: from one side, for the difficulties, both analytical and numerical, in obtaining the finite temperature equation of state for non-Abelian anyons (see the discussion in [90]); from another side, because most of the efforts have been focused in the last decade on the study of two-dimensional systems which are gapped in the bulk and gapless on the edges, as for the states commonly studied for the fractional quantum Hall effect, while, on the contrary, the two-dimensional free gas of anyons is gapless. However, there is by now a mounting interest in the study of three-dimensional topological insulators, systems gapped in the bulk, but having protected conducting gapless states on their edge or surface [31]: exotic states can occur at the surface of a three-dimensional topological insulator due to an induced energy gap, and a superconducting energy gap leads to a state supporting Majorana fermions, providing new possibilities for the realization of topological quantum computation. This surging of activity certainly calls for an investigation of the finite temperature properties of general gapless topological states on the two-dimensional surface of three-dimensional topological insulators and superconductors.

In this Chapter we focus on the study of the thermodynamics of an ideal gas of a general class of NACS particles in presence of general soft-core boundary conditions: explicit results are found for the second virial coefficient. Results for hard-core non-Abelian anyons, which is a limiting case of soft-core conditions, are presented too. The Chapter is structured as follows: in Section 4.2 we introduce the Verlinde's NACS model and, as an introduction to the subsequent discussion, in Section 4.2.1 we briefly recall the results for an ideal gas of hard-core Abelian anyons and we present in detail the general soft-core version of the Abelian anyonic model. The properties of the virial expansion are also reviewed and the monotonic behaviour of the second virial coefficient B_2 with respect to the statistical parameter is taken in exam as the hard-core parameter changes: we observe, in particular, that for a narrow range of the soft-core parameter B_2 can be non-monotonic. In Section 4.2.2 we define more precisely the NACS model and we explicitly present the set of soft-core parameters associated to the most general boundary conditions of the wave-functions. In Section 4.3.1 the coefficient B_2 is evaluated for a system of NACS particles with hard-core boundary conditions: we compare our results with previous determination of this quantity and we make some comments about limit cases. In Section 4.3.2 we study B_2 for non-Abelian anyons when soft-core wavefunction boundary conditions are allowed, with special attention to the case of isotropic boundary conditions. In Section 4.4 we summarize the virial expansions for the ideal gas of NACS particles endowed with general boundary conditions of the wave functions.

4.2 The Model

In this Section we introduce the Abelian and non-Abelian models which are object of the Chapter: in Section 4.2.1 we first briefly remind the well-known results for the thermodynamics of the ideal gas of hard-core Abelian anyons. The general soft-core version of the Abelian anyonic model is then introduced, and the behaviour of the second virial coefficient is studied as a function of the defined hard-core parameter. In Section 4.2.2 we define the NACS model, whose second virial coefficient will be derived and studied in the next Section.

4.2.1 Abelian Anyons

The thermodynamics for a system of identical Abelian anyons has been developed starting with the seminal paper [91], in which the exact quantum expression for the second virial coefficient is derived:

$$B_2^{h.c.}(2j+\delta,T) = -\frac{1}{4}\lambda_T^2 + |\delta|\lambda_T^2 - \frac{1}{2}\delta^2\lambda_T^2 \ . \tag{4.3}$$

Eq. (4.3) holds for an ideal gas of anyons whose wavefunction fulfills hard-core wavefunction boundary conditions. In (4.3) $\alpha = 2j+\delta$, where α represents the statistical parameter of the anyons [73], j is an integer and $|\delta| \leq 1$. We remind that $\alpha = 0$ and $\alpha = 1$ corresponds respectively to free two-dimensional spinless bosons and fermions [73]. Furthermore λ_T is the thermal wavelength defined as

$$\lambda_T = \left(\frac{2\pi\hbar^2}{Mk_BT}\right)^{1/2} \ . \tag{4.4}$$

As discussed in statistical mechanics textbooks, the virial expansion is done in powers of $\rho\lambda_T^2$ (where ρ is the density and M is the mass of the particles) and in the low-density, high-temperature regime, the second virial coefficient gives the leading contribution to the deviation of the equation of state from the non-interacting case, as a result of rewriting the grand canonical partition function as a cluster expansion [113, 114].

The virial coefficient (4.3) turns out to be a simple, periodic (with period 2) but non-analytic function of the statistical parameter α, showing cusps in correspondence of all its bosonic points $\alpha = 2j$. This quantity has been evaluated by different methods: one of them consists in an hard-disk-type regularization of the two-anyonic spectrum while another one is a based on path-integral approach yielding the two-body partition function, carried on by identifying the Lagrangian of the system with the one relative to the Bohm-Aharonov effect [115]. Eq. (4.3) is also retrieved by heat kernel methods, i.e. discretizing the two-particle spectrum through the introduction of a harmonic regulator potential and then directly considering the problem in the continuum [116]. Finally, another method to get Eq. (4.3) is to use a semi-classical method, which nevertheless produces the exact quantum result [92, 90]. Exact results for higher virial coefficients are not known, but a fair amount of information is available both for the third virial coefficient and for higher virial coefficients [73].

The expression (4.3) is the exact quantum result for the hard-core case, corresponding to impose the vanishing of the two-anyonic wavefunctions in the coincident points (the limit configurations for which the coordinates of two anyons coincide). However, any arbitrary boundary condition for the wave-function is in principle admissible: it is the comparison with results from the microscopic interacting Hamiltonian that should fix the relevant boundary conditions to be imposed. The second virial coefficient for Abelian anyons in this general case has been studied in [102, 117, 103].

By relaxing the regularity requirements, allowing wave-functions to diverge for vanishing relative distance r between the anyons according to the method of self-adjoint extensions, it is possible to obtain a one-parameter family of boundary conditions. The hard-core limit corresponds to scale-invariance in a field theoretical approach [118, 119, 101, 120, 121], where the scale can be precisely related to the hard-core parameter that will be defined below. The study of B_2 shows that the results for hard-core case are rather peculiar: for instance, the cusps at the bosonic points of the hard-core case are a special feature of the scale-invariant limit, which is however absent for all the soft-core cases. On the contrary, cusps are generated at all the fermionic points for all the wavefunction boundary conditions, except just for the hard-core case.

The *relative* two-body Hamiltonian for a free system of anyons with statistical parameter α, written in the bosonic description, is of the form [73]

$$H_{rel} = \frac{1}{M}(\vec{p} - \alpha\vec{A})^2 \ , \tag{4.5}$$

where $\vec{A} = (A^1, A^2)$ and $A^i \equiv \frac{\epsilon^{ij} x^j}{r^2}$ ($i = 1, 2$ and ϵ^{ij} is the completely antisymmetric tensor). The corresponding single-particle partition function of the relative dynamics is $Z_{rel} = \mathrm{Tr} e^{-\beta H_{rel}}$, where $\beta = 1/k_B T$. In order to proceed with the choice of a given self-adjoint extension, one has to define the space over which the trace above is performed. If we consider the radial component R_n of the relative wave-function ψ, the Schrödinger equation takes the form

$$\frac{1}{M}\left[-\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}+\frac{(n+\alpha)^2}{r^2}\right]R_n(r)=ER_n(r)\equiv\frac{k^2}{M}R_n(r)\ , \tag{4.6}$$

with n even (choosing the bosonic description). Without any loss of generality, the statistical parameter can be chosen as $\alpha\in[-1,1]$. Eq. (4.6) is the Bessel equation and its general solution is given in terms of the Bessel functions

$$R_n(r)=AJ_{|n+\alpha|}(kr)+BJ_{-|n+\alpha|}(kr)\ . \tag{4.7}$$

For $n\neq 0$ the constant B must vanish in order to satisfy the normalization of the relative wave-function, while for $n=0$ (s-wave) arbitrary constants A, B are allowed. This yields a one-parameter family of boundary conditions for the s-wave solution:

$$R_0(r)=(const.)\left[J_{|\alpha|}(kr)+\sigma\left(\frac{k}{\kappa}\right)^{2|\alpha|}J_{-|\alpha|}(kr)\right]\ , \tag{4.8}$$

where $\sigma=\pm 1$ and κ is a scale introduced by the boundary condition.

We will refer to

$$\varepsilon\equiv\frac{\beta\kappa^2}{M} \tag{4.9}$$

as the *hard-core parameter* of the gas. For $\varepsilon\to\infty$ with $\sigma=+1$ we retrieve the hard-core case ($\psi(0)=0$). If $\sigma=-1$, in addition to the solution (4.8), there is a bound state with energy $E_B=-\varepsilon k_BT=-\kappa^2/M$ and wavefunction

$$R_0(r)=(const.)\,K_{|\alpha|}(\kappa r)\ . \tag{4.10}$$

By proceeding as in [91], and observing that only the s-wave energy spectrum is modified with respect to the hard-core case, one gets that the second virial coefficient for a generic soft-core is given by

$$B_2^{s.c.}(T)=B_2^{h.c.}(T)-2\lambda_T^2\left\{e^{-\beta E_B}\theta(-\sigma)+\lim_{R\to\infty}\sum_{s=0}^{\infty}\left[e^{-\beta\frac{\tilde{k}_s^2}{M}}-e^{-\beta\frac{k_{0,s}^2}{M}}\right]\right\}\ , \tag{4.11}$$

where $\theta(x)$ is the Heaviside step function, $k_{0,s}R$ is the s-th zero of $J_{|\alpha|}(kR)=0$, $\tilde{k}_sR$ is the s-th zero of (4.8), and $B_2^{h.c.}$ is the hard-core result (4.3). It is possible to rewrite Eq. (4.11) in an integral form [103] as

$$B_2^{s.c.}(T)=B_2^{h.c.}(T)-2\lambda_T^2\left\{e^{\varepsilon}\theta(-\sigma)+\frac{\alpha\sigma}{\pi}\sin\pi\alpha\int_0^{\infty}\frac{dte^{-\varepsilon t}t^{|\alpha|-1}}{1+2\sigma\cos\pi\alpha\ t^{|\alpha|}+t^{2|\alpha|}}\right\}\ . \tag{4.12}$$

For $\varepsilon \gg 1$ one gets

$$B_2^{s.c.}(T) = B_2^{h.c.}(T) - 2\lambda_T^2 \left[e^{\varepsilon}\theta(-\sigma) + \frac{\sigma}{\pi}\frac{\Gamma(|\alpha|+1)}{\varepsilon^{|\alpha|}} \sin \pi|\alpha| + \cdots\right] \quad (4.13)$$

while for $\varepsilon \ll 1$

$$B_2^{s.c.}(T) = B_2^{h.c.}(T) - 2\lambda_T^2|\alpha| \left(1 - \sigma\, \varepsilon^{|\alpha|} + \cdots\right) \ . \quad (4.14)$$

Near the bosonic point $\alpha = 0$ one has for $\varepsilon \neq 0$:

$$B_2^{s.c.}(T) = -\left[\frac{1}{4} + 2\nu(\varepsilon)\theta(-\sigma)\right] \lambda_T^2 + O(\alpha^2) \ ,$$

where $\nu(\varepsilon)$ is the Neumann function defined by

$$\nu(\varepsilon) = \int_0^\infty \frac{dt\, \varepsilon^t}{\Gamma(t+1)} \ .$$

From this expression one sees that $B_2^{s.c.}(T)$ is a smooth function of the statistical parameter near $\alpha = 0$. On the contrary, near the fermionic point $|\alpha| = 1$ one has

$$B_2^{s.c.}(T) = \left[-\frac{1}{2}(1-|\alpha|)^2 + \frac{1}{4} - 2e^{-\sigma\varepsilon}\right] \lambda_T^2 + f_\sigma(\varepsilon)(1-|\alpha|) + \cdots \ ,$$

where $f_\sigma(\varepsilon)$ ($\sigma = \pm 1$) are functions of ε [not reported here], so that in general the soft-core case presents a cusp at $|\alpha| = 1$.

The virial coefficient $B_2^{s.c.}$ is plotted in Fig.4.1 for some values of the hard-core parameter ε. The plot of the second virial coefficient clearly exhibits its smoothing in the bosonic points and its sharpening in the fermionic ones, as soon as the hard-core condition is relaxed. We observe that the restriction of $B_2^{s.c.}$ over the interval $[0, 1]$ is not a monotonic function of α for each ε. This not-monotonicity happens around $\varepsilon \sim 1.4$ in a narrow range of values of ε ($1.344 < \varepsilon < 1.526$), as emphasized in Fig.4.2.

4.2.2 Non-Abelian Anyons

The main part of the present Chapter deals with the study of the low-density statistical mechanics properties of a two-dimensional gas of $SU(2)$ non-Abelian Chern-Simons (NACS) spinless particles. The NACS particles are pointlike sources mutually interacting via a topological non-Abelian Aharonov-Bohm effect [89]. These particles carry non-Abelian charges and

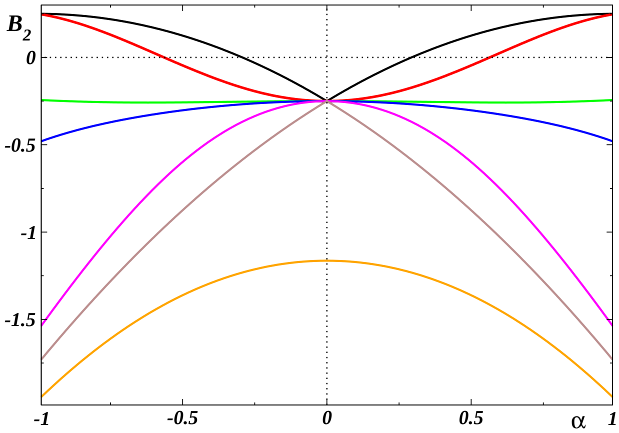

Figure 4.1: B_2 (in units of λ_T^2) vs. the statistical parameter α for different values of the hard-core parameter ε for Abelian anyons. The 6 upper curves are obtained for $\sigma = 1$: from top to the bottom ε takes the values ∞ (hard-core), 10, 1.4, 1, 0.1 and 0. The curve below all those is obtained for $\sigma = -1$ and $\varepsilon = 0.1$ and it has a shifted value at the bosonic points (the dotted lines just denote the x and y axes). We remind that B_2 is periodic in α with period 2, so we plot it in the interval $[-1, 1]$. B_2 is symmetric with respect to all the integer value of α: we see from the figure the difference in the position of the cusps of the patterns, between the hard-core and the soft-core cases.

non-Abelian magnetic fluxes, so that they acquire fractional spins and obey braid statistics as non-Abelian anyons.

In order to proceed with the computation of the second virial coefficient of a free gas of NACS particles, we first introduce the NACS quantum mechanics [122, 38, 123, 124, 104] considering the general frame of soft-core NACS particles [105, 106]. The Hamiltonian describing the dynamics of the N-body system of free NACS particles can be derived by a Lagrangian with a Chern-Simons term and a matter field coupled with the Chern-Simons gauge term [104]: the resulting Hamiltonian reads

$$H_N = -\sum_{\alpha=1}^{N} \frac{1}{M_\alpha} \left(\nabla_{\bar{z}_\alpha} \nabla_{z_\alpha} + \nabla_{z_\alpha} \nabla_{\bar{z}_\alpha} \right) \tag{4.15}$$

where M_α is the mass of the α-th particles, $\nabla_{\bar{z}_\alpha} = \frac{\partial}{\partial \bar{z}_\alpha}$ and

$$\nabla_{z_\alpha} = \frac{\partial}{\partial z_\alpha} + \frac{1}{2\pi\kappa} \sum_{\beta \neq \alpha} \hat{Q}_\alpha^a \hat{Q}_\beta^a \frac{1}{z_\alpha - z_\beta} \quad .$$

In Eq. (4.15) $\alpha = 1, \ldots, N$ labels the particles, $(x_\alpha, y_\alpha) = (z_\alpha + \bar{z}_\alpha, -i(z_\alpha - \bar{z}_\alpha))/2$ are their spatial coordinates, and $\hat{Q}^a$'s are the isovector operators in a representation of isospin l. The quantum number l labels the irreducible representations of the group of the rotations induced by the coupling of the NACS particle matter field with the non-Abelian gauge field: as a consequence, the values of l are of course quantized and vary over all the integer and the half-integer numbers, with $l = 1/2$ being the smaller possible non-trivial value ($l = 0$ corresponds to a system of free bosons). As usual, a basis of isospin eigenstates can be labeled by l and the magnetic quantum number m (varying in the range $-l, -l+1, \cdots, l-1, l$).

The virial coefficients then depend in general on the value of the isospin quantum number l and on the coupling κ (and of course on the temperature T). The quantity κ in (4.15) is a parameter of the theory. In order to enforce the gauge covariance of the theory the condition $4\pi\kappa =$ integer has to be satisfied. In the following we denote for simplicity the integer $4\pi\kappa$ by k:

$$4\pi\kappa \equiv k \ . \tag{4.16}$$

The physical meaning of κ in the NACS model can be understood removing the interaction terms in H_N by a similarity transformation:

$$\begin{aligned} H_N &\longrightarrow UH_NU^{-1} = H_N^{\mathrm{free}} = -\sum_{\alpha}^{N} \frac{2}{M_\alpha} \partial_{\bar{z}_\alpha} \partial_{z_\alpha} \\ \Psi_H &\longrightarrow U\Psi_H = \Psi_A \end{aligned} \tag{4.17}$$

where $U(z_1, \ldots, z_N)$ satisfies the Knizhnik-Zamolodchikov (KZ) equation [39]

$$\left(\frac{\partial}{\partial z_\alpha} - \frac{1}{2\pi\kappa} \sum_{\beta \neq \alpha} \hat{Q}^a_\alpha \hat{Q}^a_\beta \frac{1}{z_\alpha - z_\beta} \right) U(z_1, \ldots, z_N) = 0 \ , \tag{4.18}$$

and $\Psi_H(z_1, \ldots, z_N)$ stands for the wavefunction of the N-body system of the NACS particles in the holomorphic gauge. A comparison between the last equation and the KZ equation satisfied by the Green's function in the conformal field theory shows that $(4\pi\kappa - 2)$ corresponds to the level of the underlying $SU(2)$ current algebra. In [123] it is shown how $\Psi_A(z_1, \ldots, z_N)$ obeys the braid statistics due to the transformation function $U(z_1, \ldots, z_N)$, while $\Psi_H(z_1, \ldots, z_N)$ satisfies ordinary statistics: $\Psi_A(z_1, \ldots, z_N)$ can be then referred to as the NACS particle wavefunction in the anyon gauge.

The statistical mechanics of the NACS particles can be studied by introducing the grand partition function Ξ, defined in terms of the $N-$body

Hamiltonian H_N and the fugacity ν as

$$\Xi = \sum_{N=0}^{\infty} \nu^N \operatorname{Tr} e^{-\beta H_N} \quad . \tag{4.19}$$

In the low-density regime, a cluster expansion can be applied to Ξ [113, 114]:

$$\Xi = \exp\left(V \sum_{n=1}^{\infty} b_n \nu^n \right) \quad , \tag{4.20}$$

where V is the volume of the gas (of course, for a two-dimensional gas V equals the area A) and b_n is the n-th cluster integral, with

$$b_1 = \frac{1}{A} Z_1, \quad b_2 = \frac{1}{A}\left(Z_2 - \frac{Z_1^2}{2} \right) \tag{4.21}$$

and $Z_N = \operatorname{Tr} e^{-\beta H_N}$ being the N-particle partition function.

The virial expansion (i.e. the pressure expressed in powers of the density $\rho = \frac{N}{A}$) is given as

$$P = \rho k_B T \left[1 + B_2(T)\rho + B_3(T)\rho^2 + \ldots \right] \quad , \tag{4.22}$$

where $B_n(T)$ is the n-th virial coefficient. The second virial coefficient $B_2(T)$ is written as

$$B_2(T) = -\frac{b_2}{b_1^2} = A\left(\frac{1}{2} - \frac{Z_2}{Z_1^2} \right) \quad . \tag{4.23}$$

We assume that the NACS particles belong to the same isospin multiplet $\{|l, m>\}$ with $m = -l, \ldots, l$. The quantity $Z_1 = \operatorname{Tr} e^{-\beta H_1}$ is then given by

$$Z_1 = (2l+1) A / \lambda_T^2 \quad . \tag{4.24}$$

The computation of $Z_2 = \operatorname{Tr} e^{-\beta H_2}$ is discussed in [110], where the results for the hard-core case are presented. It is convenient to separate the center-of-mass and relative coordinates: defining $Z = (z_1 + z_2)/2$ and $z = z_1 - z_2$ one can write

$$H_2 = H_{\rm cm} + H_{\rm rel} = -\frac{1}{2\mu} \partial_Z \partial_{\bar{Z}} - \frac{1}{\mu} (\nabla_z \nabla_{\bar{z}} + \nabla_{\bar{z}} \nabla_z) \quad , \tag{4.25}$$

where $\mu \equiv M/2$ is the two-body reduced mass, $\nabla_{\bar{z}} = \partial_{\bar{z}}$ and

$$\nabla_z = \partial_z + \frac{\Omega}{z} \quad .$$

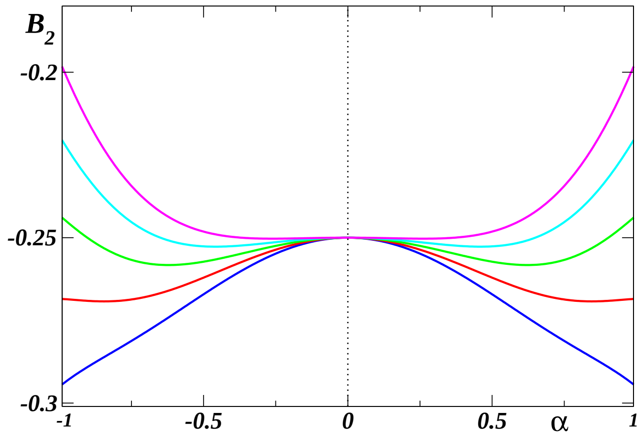

Figure 4.2: The narrow range of values of the hard-core parameter ε within which the second virial coefficient B_2 (for Abelian anyons) is a non-monotonous function of α in the interval $[0, 1]$. The parameters are $\sigma = +1$ for all the curves and $\varepsilon = 1.50,\ 1.45,\ 1.40,\ 1.35,\ 1.30$ from top to bottom (B_2 is expressed in units of λ_T^2).

Ω is a block-diagonal matrix given by

$$\Omega = \hat{Q}_1^a \hat{Q}_2^a / (2\pi\kappa) = \sum_{j=0}^{2l} \omega_j \otimes I_j \ ,$$

with $\omega_j \equiv \frac{1}{4\pi\kappa}\left[j(j+1) - 2l(l+1)\right]$.

Z_2 can be then written as

$$Z_2 = 2A\lambda_T^{-2} Z_2' \ , \tag{4.26}$$

where $Z_2' = \mathrm{Tr}_{\mathrm{rel}}\, e^{-\beta H_{\mathrm{rel}}}$. The similarity transformation $G(z, \bar{z}) = \exp\left\{-\frac{\Omega}{2}\ln(z\bar{z})\right\}$, acting as

$$\begin{aligned} H_{\mathrm{rel}} &\longrightarrow H_{\mathrm{rel}}' = G^{-1} H_{\mathrm{rel}} G, \\ \Psi(z, \bar{z}) &\longrightarrow \Psi'(z, \bar{z}) = G^{-1}\Psi(z, \bar{z}) \end{aligned} \tag{4.27}$$

gives rise to an Hamiltonian H_{rel}' manifestly Hermitian and leaves invariant Z_2'. The explicit expression for H_{rel}' is

$$H_{\mathrm{rel}}' = -\frac{1}{\mu}\left(\nabla_z' \nabla_{\bar{z}}' + \nabla_{\bar{z}}' \nabla_z'\right) \ , \tag{4.28}$$

where $\nabla'_z = \partial_z + \Omega/2z$ and $\nabla'_{\bar{z}} = \partial_{\bar{z}} - \Omega/2\bar{z}$.

By rewriting $H'_{\rm rel}$ in polar coordinates and projecting it onto the subspace of total isospin j, its correspondence with the Hamiltonian for (Abelian) anyons in the Coulomb gauge, having statistical parameter given by $\alpha_s = \omega_j$, becomes evident:

$$H'_j = -\frac{1}{2\mu}\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{\partial}{\partial\theta} + i\omega_j\right)^2\right] \ . \qquad (4.29)$$

The same analysis discussed in Section 4.2.1 shows that the radial factor of the j, j_z- component of the relative $(2l+1)^2-$vector wavefunction $\psi = e^{in\theta}R_n(r)$ obeys the Bessel equation

$$\frac{1}{M}\left[-\frac{1}{r}\frac{d}{dr}r\frac{d}{dr} + \frac{(n+\omega_j)^2}{r^2}\right]R_n^{j,j_z}(r) = ER_n^{j,j_z}(r) \equiv \frac{k^2}{M}R_n^{j,j_z}(r) \ , \qquad (4.30)$$

whose general solution is

$$R_n^{j,j_z}(r) = A^{j,j_z}J_{|n+\omega_j|}(kr) + B^{j,j_z}J_{-|n+\omega_j|}(kr) \ . \qquad (4.31)$$

As already discussed in the previous Section 4.2.1, B^{j,j_z} can be nonzero only in the case $n = 0$ ($s-$wave). Then the $s-$wave gives rise to a one-parameter family of boundary conditions

$$R_0^{j,j_z}(r) = (\text{const.})\left[J_{|\omega_j|}(kr) + \sigma\left(\frac{k}{\kappa_{j,j_z}}\right)^{2|\omega_j|}J_{-|\omega_j|}(kr) \ ,\right] , \qquad (4.32)$$

where $\sigma = \pm 1$, and κ_{j,j_z} is a momentum scale introduced by the boundary condition.

We refer to the $(2l+1)^2$ quantities

$$\varepsilon_{j,j_z} \equiv \frac{\beta\kappa^2_{j,j_z}}{M} \qquad (4.33)$$

as *hard-core parameters* of the system. The hard-core limit corresponds to $\varepsilon_{j,j_z} \to \infty$ for all j, j_z.

We conclude this Section by observing that, according to the regularization used in [91, 116], the second virial coefficient is defined as

$$B_2(\kappa, l, T) - B_2^{(n.i.)}(l, T) = -\frac{2\lambda_T^2}{(2l+1)^2}\left[Z'_2(\kappa, l, T) - Z_2'^{(n.i.)}(l, T)\right] \ , \qquad (4.34)$$

where $B_2^{(n.i.)}(l, T)$ is the virial coefficient for the system with particle isospin l and without interaction ($\kappa \to \infty$), which will be expressed in terms of the virial coefficients $B_2^B(T)$, $B_2^F(T)$ of the free Bose and Fermi systems with the considered general wavefunction boundary conditions. Furthermore, $Z_2'(\kappa, l, T) - Z_2'^{(n.i.)}(l, T)$ is the (convergent) variation of the divergent partition function for the two-body relative Hamiltonian, between the interacting case in exam and the non-interacting limit ($\kappa \to \infty$).

4.3 Second Virial Coefficient

In this Section we present our results for the second virial coefficient of a free gas of NACS particles: we will first study the hard-core case in Section 4.3.1, comparing in detail our findings with results available in literature [109, 110, 111, 112]. We then study B_2 for non-Abelian anyons when general soft-core wavefunction boundary conditions, focusing the attention in particular to the isotropic boundary conditions.

4.3.1 Hard-Core Case

The hard-core case is obtained in the limit $\varepsilon_{j,j_z} \to \infty$ for all j, j_z. The second virial coefficient has been discussed in literature, and different results for B_2 have been presented [109, 110, 111]: the differences between such results have been discussed, see in particular Ref. [110] and the comment [111]. Our findings differ from results presented in [109, 110, 111]: in this Section, as well as in Appendices 5.3.2-5.3.2, a detailed comparison with such available results will be presented.

For hard-core boundary conditions of the relative two-anyonic vectorial wavefunction, the quantity $B_2^{(n.i.)}$ entering Eq. (4.34) is found to be [110]

$$B_2^{(n.i.)}(l, T) = \frac{1}{(2l+1)^2} \sum_{j=0}^{2l} (2j+1) \left[\frac{1+(-1)^{j+2l}}{2} B_2^B(T) + \frac{1-(-1)^{j+2l}}{2} B_2^F(T) \right] \quad (4.35)$$

Using for the hard-core case the known values $B_2^B(T) = -B_2^F(T) = -\frac{1}{4}\lambda_T^2$ [90], one then obtains

$$B_2^{(n.i.)}(l, T) = -\frac{\lambda_T^2}{4} \frac{1}{2l+1} \quad . \quad (4.36)$$

To proceed further, we introduce a regularizing harmonic potential $\mathcal{V} = \frac{\mu}{2}\epsilon^2 r^2$ [91, 90], whose effect is to make discrete the spectrum of H_j'. Using the notations of [73] (see pg.48), the spectrum consists of the following two classes: $E_n^I = \epsilon(2n+1+\gamma_j)$ with degeneracy $(n+1)$, and $E_n^{II} = \epsilon(2n+1-\gamma_j)$

with degeneracy n, where n is a non-negative integer and $\gamma_j \equiv \omega_j \, mod\, 2$. It follows that the regularized partition function reads

$$Z_2'(\kappa, l, T) - Z_2'^{(n.i.)}(l, T) = \sum_{j=0}^{2l}(2j+1)\lim_{\epsilon\to 0}\left\{\frac{1+(-1)^{j+2l}}{2}\left\{Z_\epsilon'(\gamma_j) - Z_\epsilon'(0)\right\} + \right.$$

$$\left. +\frac{1-(-1)^{j+2l}}{2}\left\{Z_\epsilon'[(\gamma_j+1)\, mod\, 2] - Z_\epsilon'(1)\right\}\right\} , \tag{4.37}$$

with

$$Z_\epsilon'(\gamma_j) = \sum_{n=0}^{\infty}\left[(n+1)\, e^{-\beta\epsilon(2n+1+\gamma_j)} + n\, e^{-\beta\epsilon(2n+1-\gamma_j)}\right] = \frac{1}{2}\frac{\cosh\left[\beta\epsilon(\gamma_j - 1)\right]}{\sinh^2\beta\epsilon} \quad .$$

The final result for the NACS gas in the hard-core limit is then the following:

$$B_2^{h.c.}(\kappa, l, T) = -\frac{\lambda_T^2}{4}\frac{1}{2l+1} +$$

$$-\frac{\lambda_T^2}{2(2l+1)^2}\sum_{j=0}^{2l}(2j+1)\left[\frac{1+(-1)^{j+2l}}{2}(\gamma_j^2 - 2\gamma_j) + \frac{1-(-1)^{j+2l}}{2}\left[(\gamma_j+1)\, mod\, 2 - 1\right]^2\right] \quad . \tag{4.38}$$

Eq. (4.38) is the main result of this Subsection: the dependence of B_2 on k for some fixed l's, and vice versa on l for some fixed k's is represented in Figs.4.3-4.4: one sees from Fig.4.3 a non-monotonic behavior of B_2 as a function of k. In Fig.4.4 one can see that the values of B_2 vs. l form two different groups, depending on integer and half-integer values of l.

As a first check of Eq. (4.38), we observe that the value of $B_2^{(n.i.)}(l, T)$ of the free case (corresponding to the limit $1/4\pi\kappa \to 0$) is correctly reproduced: indeed, for a given l one has in this limit $\omega_j \to 0^\pm$, $\gamma_j \to \left\{\begin{array}{l} 0^+ \\ 2^- \end{array}\right.$, $\gamma_j^2 - 2\gamma_j \to 0^-$, $[(\gamma_j+1)\, mod\, 2 - 1]^2 \to 0^+$, as it might. A more detailed discussion on the limit $1/4\pi\kappa \to 0$ is presented in Appendix 5.3.2.

To further compare with available results, we observe that in [110] and [112] it was stated that the factors $(-1)^{2l}$ should not appear in the expression of the two-particle partition function, or in the expression of the second virial coefficient (l denoting the isospin quantum number of each particle). However, as pointed out in [111] and as further motivated in the following, such factors are needed. To clarify this issue it is convenient to make reference to the properties of the Clebsch-Gordan coefficients. The two (spinless)

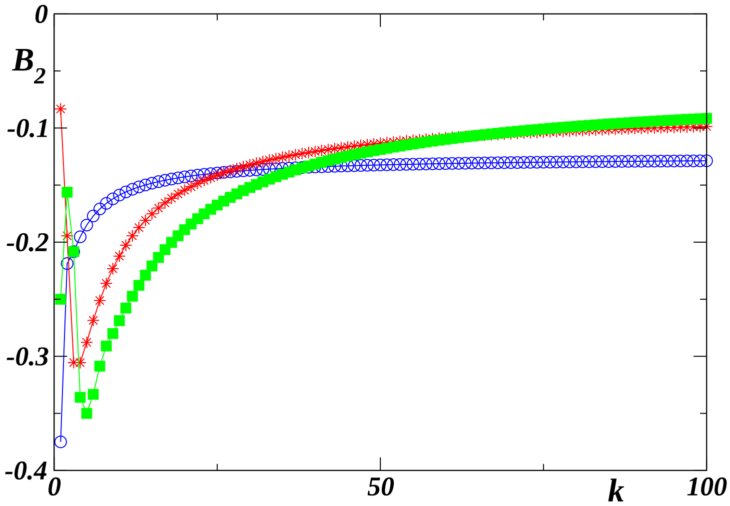

Figure 4.3: B_2 vs. $k = 4\pi\kappa$ for an hard-core NACS gas with $l = 1/2$ (blue open circles), $l = 1$ (red stars), $l = 3/2$ (green squares). In this and the following figures B_2 is in units of λ_T^2 (furthermore the line is just a guide for the eye, since k assumes only integer values). The values of B_2 in the limit $\kappa \to \infty$ are given by $-\frac{1}{8}, -\frac{1}{12}, -\frac{1}{16}$ and are correctly reproduced.

particles in exam have both isospin l and total isospin j: the Clebsch-Gordan coefficients express the change of basis, in the two-body isospin space, between the basis labeled by the individual magnetic isospin numbers m_1, m_2 and the basis labeled by the total and magnetic isospin j, m_j:

$$|lljm_j\rangle = \sum_{m_1=-l}^{l} \sum_{m_2=-l}^{l} \langle lm_1 lm_2 | jm_j\rangle |lm_1 lm_2\rangle \ ,$$

where the Clebsch-Gordan coefficients fulfill the symmetry property

$$\langle lm_1 lm_2 | jm_j\rangle = (-1)^{2l-j} \langle lm_2 lm_1 | jm_j\rangle \ .$$

Notice that the real spin of the particles is not taken in account in this consideration. With respect to the exchange of all the quantum numbers, the isospin two-body wavefunction corresponding to a state of total isospin j takes a factor $(-1)^{2l-j} = (-1)^{j+2l}$ (being j an integer), so that the factor $\frac{1+(-1)^{j+2l}}{2}$ projects over the states for which the partition function can be evaluated in the bosonic basis, the factor $\frac{1-(-1)^{j+2l}}{2}$ projects over those for which the partition function can be evaluated in the fermionic basis, from which Eqs. (4.35), (4.36) and (4.38) can be obtained.

Our result (4.38) differs also from the results presented in [109], where a method of computation of the second virial coefficient for hard-core NACS gas based on the idea of averaging over all the isospin states is proposed. In particular, in [109] the special cases $l = 1/2$, $l = 1$, and the large-κ limit for two particles belonging to a representation l with $\lim_{l\to\infty} \frac{l^2}{4\pi\kappa} = a < 1$ were considered. In the last limit the sum over all the resulting total isospins $r \leq 2l$ is approximated by an integral. The results are given by Eqs. (35),(36) and (38) of [109]:

$$\begin{aligned} B_2^{h.c.}(k, l = 1/2, T) &= \lambda_T^2 \left(-\frac{1}{4} + \frac{3}{4k} - \frac{3}{8k^2} \right) \ , \\ B_2^{h.c.}(k, l = 1, T) &= \lambda_T^2 \left(-\frac{1}{4} + \frac{20}{9k} - \frac{8}{3k^2} \right) \quad \forall k > 1 \ , \\ \lim_{l\to\infty} B_2^{h.c.}(k, l, T) &= \lambda_T^2 \left(-\frac{1}{4} + a - \frac{a^2}{3} \right) \ , \end{aligned} \tag{4.39}$$

while our corresponding results are

$$\begin{aligned} B_2^{h.c.}(k, l = 1/2, T) &= \lambda_T^2 \left(-\frac{1}{8} + \frac{3}{8k} - \frac{3}{8k^2} \right) \quad \forall k \geq 2 \ , \\ B_2^{h.c.}(k, l = 1, T) &= \lambda_T^2 \left(-\frac{1}{12} + \frac{14}{9k} - \frac{8}{3k^2} \right) \quad \forall k \geq 4 \ , \\ \lim_{l\to\infty} B_2^{h.c.}(k, l, T) &= \lambda_T^2 \left(-\frac{a}{4} - \frac{2}{3}a^2 \right) \ . \end{aligned} \tag{4.40}$$

The limits $l \to \infty$ in (4.39) and (4.40) are taken together with $\lim_{l\to\infty} l^2/k = a$, where a is kept fixed and $a < 1$ in (4.39) and $a < 1/2$ in (4.40). The derivation from Eq. (4.38) of the three special cases above is reported in the Appendix 5.3.2. We notice that the asymptotic value found for B_2 in the third case is expected to vanish for $a = 0$, while on the contrary this does not occur for the results of [109]: indeed, $a = 0$ corresponds to consider the limit $B_2^{(n.i.)}(l, T) = \frac{-1}{2l+1} \frac{\lambda_T^2}{4}$, which vanishes in the large l-limit.

The difference between the results of Ref. [109] and ours stands in a different averaging: while in [109] the virial coefficients are expressed as averages of the virial coefficients over the $(2l+1)^2$ two-body states of isospin, in our case we take into account the effect of the isospin symmetry factor $(-1)^{j+2l}$ characterizing the states of total isospin j.

Notice that Eq. (4.38) for the second virial coefficient can be recovered using the approach presented in [111], as shown in Appendix 5.3.2. Indeed,

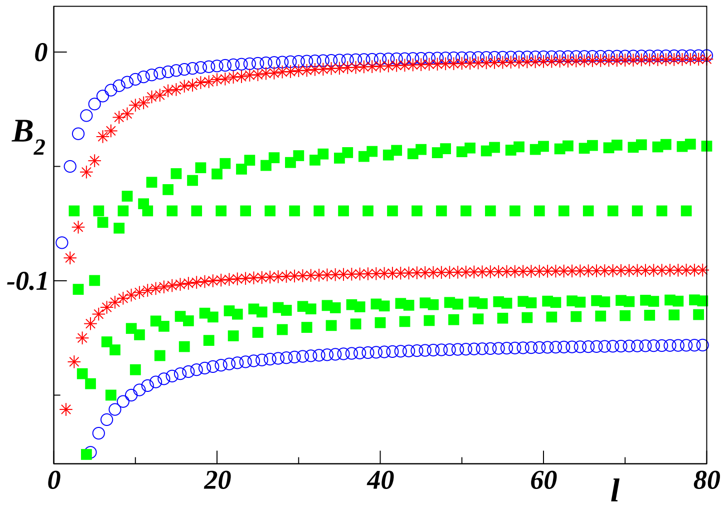

Figure 4.4: B_2 vs. l for an hard-core NACS gas with $k = 1$ (blue open circles), $k = 2$ (red stars), $k = 3$ (green squares). l varies over all the integer and the half-integer numbers: in the upper (lower) part of the figure the plotted values of B_2 correspond to integer (half-integer) values of l.

one can find from Eqs. (4.34)-(4.36)-(4.38)(see Appendix 5.3.2 for details) that the following expression for $B_2^{h.c.}$ given in Eq. (2) of [111] holds:

$$B_2^{h.c.}(\kappa, l, T) = \frac{1}{(2l+1)^2} \sum_{j=0}^{2l} (2j+1) \left[\frac{1 + (-1)^{j+2l}}{2} B_2^B(\omega_j, T) + \frac{1 - (-1)^{j+2l}}{2} B_2^F(\omega_j, T) \right] , \tag{4.41}$$

where $B_2^{B,F}(\omega, T)$ is given by [91]

$$B_2^{B(F)}(\omega, T) = \frac{1}{4} \lambda_T^2 \begin{cases} -1 + 4\delta - 2\delta^2, & \text{N even (odd)} \\ 1 - 2\delta^2, & \text{N odd (even)} \end{cases} \tag{4.42}$$

($\omega = N + \delta$ and N an integer such that $0 \leq \delta < 1$). The computation presented in Appendix 5.3.2 shows that Eq. (2) of [111] is a correct starting point to study $B_2^{h.c.}$: however, notice that Eq. (3) of [111] should be replaced with Eq. (5.51) given in Appendix 5.3.2.

We finally discuss in more detail the non-interacting limit $1/4\pi\kappa \to 0$ in order to clarify the meaning of Eq. (4.36). In the limit $k \to \infty$ the covariant derivatives in (4.15) trivialize and the isospin becomes just a symmetry of the Hamiltonian, resulting in a pure (isospin) degeneration $g = 2l+1$. Let Ξ be the grand partition function, ϵ the generic single-particle energy level for an assigned spectral discretization, and using the upper/lower signs respectively for the g-degenerate bosonic/fermionic single-particle states: notice that any value of g is allowed both in the bosonic and the fermionic case, since the statistics is not constrained by isospin. The expressions for the grand partition function, the pressure and the density are

$$\Xi(z, A, T) = \prod_{\epsilon} (1 \mp z e^{-\beta\epsilon})^{\mp g} \quad ,$$

$$\frac{PA}{k_B T} = \ln \Xi(z, A, T) = \mp \sum_{\epsilon} g \ln(1 \mp z e^{-\beta\epsilon})$$

and

$$N \equiv z \frac{\partial}{\partial z} \ln \Xi(z, A, T) = g \sum_{\epsilon} \frac{1}{z^{-1} e^{\beta\epsilon} \mp 1} \quad .$$

It follows

$$\begin{cases} \frac{(P/g)A}{k_B T} = \mp \sum_{\epsilon} \ln(1 \mp z e^{-\beta\epsilon}) \\ (\rho/g) = \sum_{\epsilon} \frac{1/A}{z^{-1} e^{\beta\epsilon} \mp 1} \end{cases} \Rightarrow \quad \frac{P}{g} = k_B T \sum_{n=0}^{\infty} \left(\frac{\rho}{g}\right)^n B_n^o \quad , \tag{4.43}$$

where B_n^o denotes the n-th virial coefficient for spinless boson(/spinless fermion) without either isospin degeneration. Hence, denoting by B_n the n-th virial coefficient in presence of isospin freedom one has

$$P = k_B T \sum_{n=0}^{\infty} \left(\rho \lambda_T^2\right)^n \frac{1}{g^{n-1}} B_n^0 = k_B T \sum_{n=0}^{\infty} \rho^n B_n$$

and therefore

$$B_n = \frac{1}{g^{n-1}} B_n^0 \quad :$$

therefore $B_2 = \frac{1}{2l+1} B_2^0$, that is exactly what is written in Eq. (4.36). The result (4.36) can be also understood by observing that all the virial coefficients for a system of NACS defined over a representation of isospin l tend, in the non-interacting limit $\kappa \to \infty$, to $(-1)^{2l}$ times those of an ideal gas of identical l-spin ordinary quantum particles. In particular, the second virial coefficient of an ideal system of quantum s-spin particles is indeed:

$$B_2(s, T) = +\frac{\lambda_T^2}{4} \frac{(-1)^{2s+1}}{2s+1} \quad , \tag{4.44}$$

in agreement with (4.36). The issue becomes much more complex for flux-carrying particles (finite κ) having a non-zero spin, as discussed in [125, 126, 127]: however, for the true ideal spinor case $\alpha = 0$ and $s = 1/2$ it is

$B_2(s = 1/2, T) = \frac{1}{8}\lambda_T^2$ [126], again in agreement with (4.44) and the related (4.36).

We conclude this Section by observing that a semiclassical computation of the second virial coefficient for a system of hard-core NACS particles reproduces Eq. (4.38): we remind that for an Abelian hard-core gas the semiclassical approximation [92, 73] yields the exact quantum result of [91] for B_2. By extending such a computation to the hard-core NACS gas we find exactly Eq. (4.38) (details are not reported here). We mention that in literature it has been conjectured that the semiclassical approximation could give the exact expressions for all the virial coefficients in presence of hard-core boundary conditions [90]: the rationale for this conjecture is that for hard-core boundary conditions there are no other length scale besides λ_T. Therefore, having established the extension to the non-Abelian hard-core case of the semiclassical computation of B_2, one could in the future obtain information about higher virial coefficients for the hard-core NACS gas. However, we alert the reader that the presence of other relevant length scales (other than λ_T) in general prevent the semiclassical approximation from being exact: an explicit example is given in [90]. We conclude that for the soft-core NACS (that we are going to treat in the next Section) the semiclassical approximation is not expected to give the correct results.

4.3.2 General Soft-Core Case

If one removes the hard-core boundary condition for the relative $(2l+1)^2$-component two-anyonic wavefunction and fixes an arbitrary external potential as a spectral regularizator, then the spectrum of each projected Hamiltonian operator H'_j can be represented as the union of the spectra of $(2j+1)$ scalar Schrödinger operators, one for each j_z-component, endowed with its respective hard-core parameter ε_{j,j_z} (as shown in Appendix 5.3.2). As discussed in Section 4.2.2, one then ends up with a set of $(2l+1)^2$ (in principle independent) parameters ε_{j,j_z}, which are needed to fix the boundary behavior. They can be organized in a $(2l+1)\times(2l+1)$ matrix:

$$\begin{pmatrix} \varepsilon_{0,0} & \varepsilon_{1,1} & \varepsilon_{2,2} & \cdots & \varepsilon_{2l+1,2l+1} \\ \varepsilon_{1,-1} & \varepsilon_{1,0} & \varepsilon_{2,1} & \cdots & \varepsilon_{2l+1,2l} \\ \varepsilon_{2,-2} & \varepsilon_{2,-1} & \varepsilon_{2,0} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \varepsilon_{2l+1,-2l-1} & \varepsilon_{2l+1,-2l} & \cdots & \cdots & \varepsilon_{2l+1,0} \end{pmatrix}. \tag{4.45}$$

Proceeding as in the previous Subsection 4.3.1, one has then for the general soft-core NACS gas the following expression for the second virial coefficient:

$$B_2^{s.c.}(\kappa,l,T) = \frac{1}{(2l+1)^2}\sum_{j=0}^{2l}\sum_{j_z=-j}^{j}\left[\frac{1+(-1)^{j+2l}}{2}B_2^B(\omega_j,T,\varepsilon_{j,j_z}) + \frac{1-(-1)^{j+2l}}{2}B_2^F(\omega_j,T,\varepsilon_{j,j_z})\right] \ , \tag{4.46}$$

where $B_2^B(\omega_j,T,\varepsilon_{j,j_z})$ is the soft-core expression entering Eq. (4.12):

$$B_2^B(\omega_j,T,\varepsilon_{j,j_z}) = B_2^{h.c.}(\delta_j,T) - 2\lambda_T^2\left\{e^{\varepsilon_{j,j_z}}\theta(-\sigma) + \frac{\delta_j\sigma}{\pi}(\sin\pi\delta_j)\int_0^\infty \frac{dte^{-\varepsilon_{j,j_z}t}t^{|\delta_j|-1}}{1+2\sigma(\cos\pi\delta_j)\,t^{|\delta_j|}+t^{2|\delta_j|}}\right\}, \tag{4.47}$$

with $\delta_j \equiv (\omega_j+1)\,mod\,2-1$, and $B_2^F(\omega_j,T,\varepsilon_{j,j_z})$ is the previous expression evaluated for $\omega_j \to \omega_j+1$:

$$B_2^F(\omega_j,T,\varepsilon_{j,j_z}) = B_2^{h.c.}(\Gamma_j,T) - 2\lambda_T^2\left\{e^{\varepsilon_{j,j_z}}\theta(-\sigma) + \frac{\Gamma_j\,\sigma}{\pi}(\sin\pi\Gamma_j)\int_0^\infty \frac{dte^{-\varepsilon_{j,j_z}t}t^{|\Gamma_j|-1}}{1+2\sigma(\cos\pi\Gamma_j)\,t^{|\Gamma_j|}+t^{2|\Gamma_j|}}\right\}, \tag{4.48}$$

with $\Gamma_j \equiv \omega_j\,mod\,2\,-1$. Eq. (4.46) is the desired result for a NACS ideal gas with general soft-core boundary conditions.

To perform explicit computations, we consider in the following the simple case in which the isotropy of the hard-core parameter is assumed within each shell with assigned isospin quantum number l. In other words, $\varepsilon_{j,j_z} \equiv \varepsilon_j$ and the matrix (4.45) then reads

$$\varepsilon_{j,j_z} \equiv \begin{pmatrix} \varepsilon_0 & \varepsilon_1 & \cdots & \varepsilon_{2l+1} \\ \varepsilon_1 & \varepsilon_1 & \cdots & \varepsilon_{2l+1} \\ \cdots & \cdots & \cdots & \cdots \\ \varepsilon_{2l+1} & \varepsilon_{2l+1} & \cdots & \varepsilon_{2l+1} \end{pmatrix} \ . \tag{4.49}$$

When all the element of the matrix (4.49) are equal, we will use the notation $\varepsilon_{j,j_z} \equiv \varepsilon$. In such a completely isotropic case, Eq. (4.46) takes the simplified form

$$B_2^{s.c.}(\kappa,l,T) = \frac{1}{(2l+1)^2}\sum_{j=0}^{2l}(2j+1)\,B_2^B(\nu_j,T,\varepsilon) \ , \tag{4.50}$$

where

$$\nu_j \equiv \left(\omega_j - \frac{1+(-1)^{j+2l}}{2}\right) mod\,2-1 \ . \tag{4.51}$$

In Figs. 4.5-4.10 we show, for three values of the isotropic hard-core parameter ε, the dependence of $B_2^{s.c.}$ on k for some fixed l's, and vice versa on l for some fixed k's. Fig.4.7 evidences that for suitable values of ε the second virial

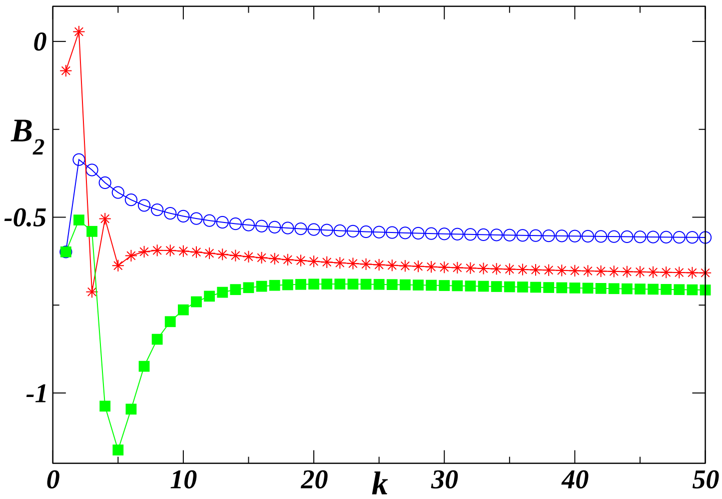

Figure 4.5: $B_2^{s.c.}(k,l,T)$ vs. k for a soft-core NACS gas with $l=1/2$ (blue open circles), $l=1$ (red stars) and $l=3/2$ (green squares): in all $\varepsilon=0.1$ (k varies over the positive integers).

coefficient may change sign and have strong variations. From Eq. (4.50) it is possible to see that the values of $B_2^{s.c.}(\kappa,l,T)$ corresponding to semi-integer l and $k=1$ are independent of l, depending only on ε and T (see Figs.4.6,4.8,4.10). In fact Eq. (4.51) yields that for l semi-integer ($l=1/2,3/2,\cdots$) and $k=1$ one has $\nu_j=\pm\frac{1}{2}$ and therefore

$$B_2^{s.c.}\left(k=1,l=n+\frac{1}{2},T\right)=B_2^B\left(\frac{1}{2},T,\varepsilon\right) \tag{4.52}$$

(with $n=0,1,2,\cdots$).

For $l=1/2$, i.e. the lowest possible value of l for non-Abelian anyons, the assumption of isotropy ($\varepsilon_{0,0}=\varepsilon_0$ and $\varepsilon_{1,m}=\varepsilon_1$ with $m=1,0,-1$) yields:

$$B_2^{s.c.}\left(\kappa,l=\frac{1}{2},T\right)=\frac{3}{4}B_2^B(\omega_1,T,\varepsilon_1)+\frac{1}{4}B_2^F(\omega_0,T,\varepsilon_0)\ . \tag{4.53}$$

As example, let us consider the case $l=1/2, 4\pi\kappa=3$:

$$B_2^{s.c.}\left(k=3,l=\frac{1}{2},T\right)=\frac{3}{4}B_2^B\left(\alpha=\frac{1}{6},T,\varepsilon_1\right)+\frac{1}{4}B_2^F\left(\alpha=-\frac{1}{2},T,\varepsilon_0\right) \tag{4.54}$$

(similar results can be found for other values of k).

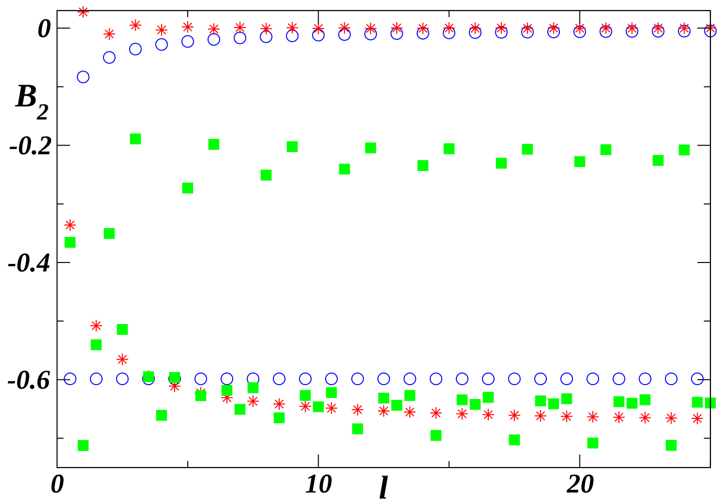

Figure 4.6: $B_2^{s.c.}(k,l,T)$ vs. l for $k=1$ (blue open circles), $k=2$ (red stars) and $k=3$ (green squares), with $\varepsilon=0.1$ (l varies over the integer and the half-integer numbers).

For the isotropic soft-core system, special boundary conditions are those limited to the s-channel (for which the p-wave is assumed to be hard-core, $\varepsilon_1=\infty$) and to the p-channel (for which, vice versa, the s-wave is assumed to be hard-core, $\varepsilon_0=\infty$). In order to assure the physical soundness of the virial expansion, $k_B T$ has to be much higher [102] than the energy of the eventual bound state E_B associated to the wavefunction (4.10). Hence, for both these channels the virial expansion is meaningful provided that we take $\sigma=+1$ in Eq. (4.12), and the virial coefficients for these two channels are

$$B_2^{s.c.}\left(k=3,l=\frac{1}{2}\right)_{s-\text{channel}} = -\frac{\lambda_T^2}{24}\left\{1+\frac{24}{\pi}\int_0^\infty dt\frac{e^{-\varepsilon_0 t}t^{-1/2}}{1+t}\right\} \tag{4.55}$$

and

$$B_2^{s.c.}\left(k=3,l=\frac{1}{2}\right)_{p-\text{channel}} = -\frac{\lambda_T^2}{24}\left\{1+\frac{4}{\pi}\int_0^\infty dt\frac{e^{-\varepsilon_1 t}t^{-5/6}}{1+\sqrt{3}\,t^{1/6}+t^{1/3}}\right\} . \tag{4.56}$$

The previous equation clearly shows that the depletion of B_2 with respect to the hard-core value $-\frac{1}{24}\lambda_T^2$ is the result of the anyonic collisions allowed by the soft-core conditions. If the four parameters of the whole matrix are taken to be identical $\varepsilon_0=\varepsilon_1\equiv\varepsilon$ (”complete isotropy” of the hard-core parameters

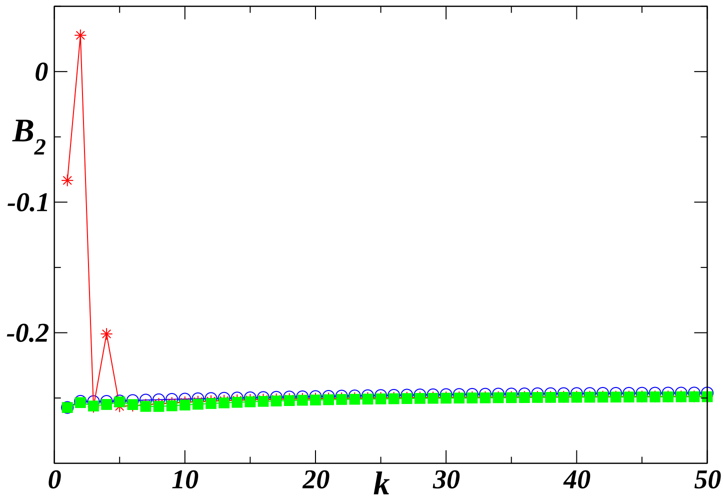

Figure 4.7: $B_2^{s.c.}(k,l,T)$ vs. k for $l = 1/2$ (blue open circles), $l = 1$ (red stars) and $l = 3/2$ (green squares), with $\varepsilon = 1.4$.

matrix), the expression for the virial coefficient reduces to

$$B_2^{(s.c.)}\left(k=3, l=\frac{1}{2}, T\right) = -\frac{\lambda_T^2}{24}\left\{1+\frac{4}{\pi}\int_0^\infty dt\, e^{-\varepsilon t}\left(\frac{6\,t^{-1/2}}{1+t}+\frac{t^{-5/6}}{1+\sqrt{3}\,t^{1/6}+t^{1/3}}\right)\right\} \quad . \tag{4.57}$$

The dependence of this quantity on ε becomes more evident by representing the ε variable in logarithmic scale, as shown in Fig.4.11. The hard-core limit value $B_2^{h.c.}/\lambda_T^2 = -1/24$ predicted by (4.38) is asymptotically approached, although for extremely high ε: e.g. for $\varepsilon = 10^{17}$ it is $B_2^{h.c.}/\lambda_T^2 \approx -0.05$, which deviates from the asymptotic value by a $\approx 20\%$. We conclude that even an extremely small deviation from the hard-core conditions may have a significant impact on B_2 and therefore on the thermodynamical properties. At variance, for small values of ε, the extension of the analysis presented in [103] for soft-core Abelian anyons allows to compute the value of B_2 for the limit case $\varepsilon = 0$: for $\varepsilon = 0$ one gets $B_2^{s.c.}\left(k=3, l=\frac{1}{2}, T\right)/\lambda_T^2 = -13/24$. The monotonically increasing behaviour of B_2 in ε is evident from (4.57), and consistent with an approach towards an hard-core (hence more repulsive) condition.

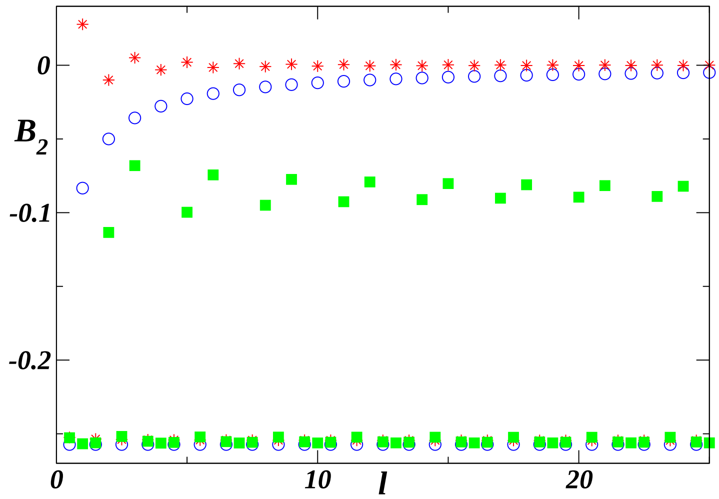

Figure 4.8: $B_2^{s.c.}(k,l,T)$ vs. l for $k=1$ (blue open circles), $k=2$ (red stars) and $k=3$ (green squares), with $\varepsilon=1.4$.

4.4 Other Thermodynamical Properties

In this Section we remind the virial expansions for the main thermodynamical quantities of the system studied in this Chapter, namely the gas of NACS particles endowed with general boundary conditions for the wavefunctions and we discuss how these quantities read at the order $\rho\lambda_T^2$ of the virial coefficient, in order to highlight the role played by the second virial coefficient B_2 computed in the previous Section.

The thermodynamical quantities are associated to the virial coefficients $\{B_n(T)\}$ of the equation of state (featuring the expansion of the pressure in powers of the number density ρ). As discussed in statistical mechanics textbooks [113, 114, 128] one has the following virial expansions for the pressure P, the Helmholtz free energy A_H, the Gibbs free energy G, the entropy S, the internal energy E and the enthalpy H (A being the area) :

$$\textit{Pressure}: \quad \frac{PA}{Nk_BT} = 1 + \sum_{k\geq 1} B_{k+1}\,\rho^k \;\; ;$$

$$\textit{Helmholtz free energy}: \quad \frac{A_H}{Nk_BT} = \log(\rho\lambda_T^2) - 1 + \sum_{k\geq 1} \frac{1}{k} B_{k+1}\,\rho^k \;\; ;$$

$$\textit{Gibbs free energy}: \quad \frac{G}{Nk_BT} = \log(\rho\lambda_T^2) + \sum_{k\geq 1} \frac{k+1}{k} B_{k+1}\,\rho^k \;\; ;$$

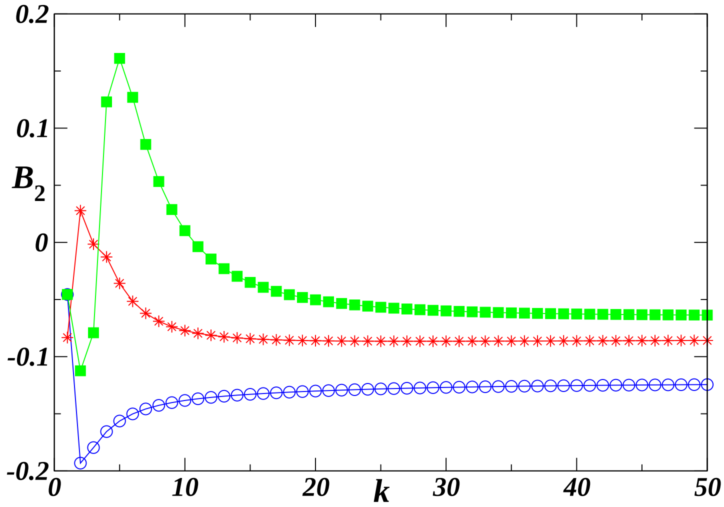

Figure 4.9: $B_2^{s.c.}(k, l, T)$ vs. k for $l = 1/2$ (blue open circles), $l = 1$ (red stars) and $l = 3/2$ (green squares), with $\varepsilon = 10$.

$$Entropy: \quad \frac{S}{Nk_B} = 2 - \log(\rho\lambda_T^2) - \sum_{k\geq 1} \frac{1}{k} \frac{\partial}{\partial T} (T\, B_{k+1})\, \rho^k \;\; ;$$

$$Internal\ energy: \quad \frac{E}{Nk_BT} = 1 - T \sum_{k\geq 1} \frac{1}{k} \frac{\partial B_{k+1}}{\partial T} \rho^k \;\; ;$$

$$Enthalpy: \quad \frac{H}{Nk_BT} = 2 + \sum_{k\geq 1} \left(B_{k+1} - \frac{1}{k} T \frac{\partial B_{k+1}}{\partial T} \right) \rho^k \;\; .$$

Using the previous expressions, stopping at the lowest order of the virial coefficient ρ (i.e. $\rho\lambda_T^2$) and using the fact that for a general NACS ideal gas one has $B_2(T) \propto T^{-1}$, one can obtain the thermodynamical quantities at the lowest order of the virial expansion: in particular we find

$$\frac{PA}{Nk_BT} = 1 + B_2\rho \;\; ;$$

$$\frac{A_H}{Nk_BT} = \log(\rho\lambda_T^2) - 1 + B_2\,\rho \;\; ;$$

$$\frac{G}{Nk_BT} = \log(\rho\lambda_T^2) + 2B_2\,\rho \;\; ;$$

$$\frac{E}{Nk_BT} = 1 + B_2\,\rho \;\; ;$$

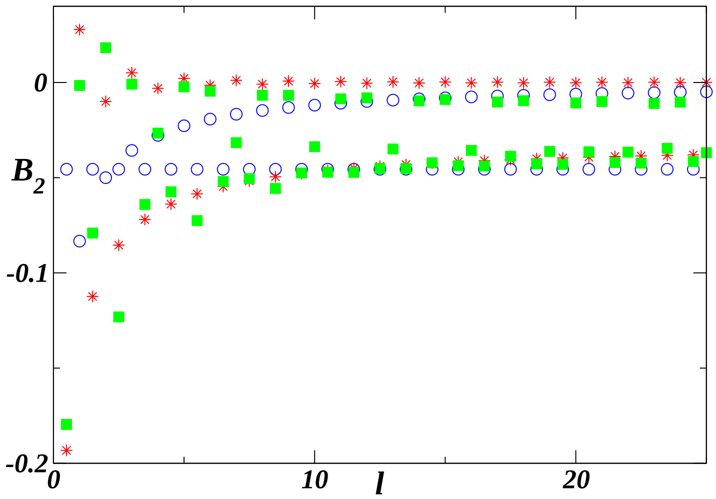

Figure 4.10: $B_2^{s.c.}(k,l,T)$ vs. l for $k=1$ (blue open circles), $k=2$ (red stars) and $k=3$ (green squares), with $\varepsilon=10$.

$$\frac{H}{Nk_BT} = 2 + 2B_2\,\rho$$

(at the lowest order of virial expansion, the entropy and the heat capacity at constant volume do not depend on B_2). Using the expression of B_2 given by Eq. (4.46) one can obtain the deviation of the various thermodynamical quantities from their ideal gas value.

An important consequence of the previous results is that at the order $\rho\lambda_T^2$ one finds

$$E = PA : \tag{4.58}$$

Eq. (4.58) is an exact identity for 2D Bose and Fermi ideal gases, valid at all the orders of the virial expansion [73]. Similarly, for the soft-core NACS ideal gas, at the order $\rho\lambda_T^2$, one has $H=2E$, which is also exact at all orders for 2D Bose and Fermi ideal gases [73]. Further investigations on the higher virial coefficients are needed to ascertain if the equation of state (4.58) is exact (at all orders) for a general soft-core NACS ideal gas.

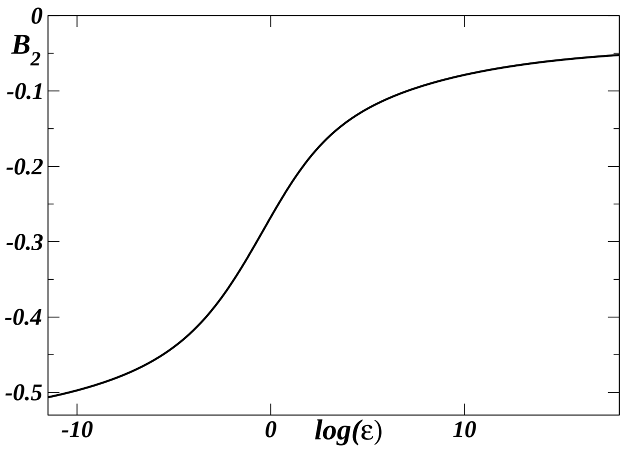

Figure 4.11: $B_2^{s.c.}$ as a function of the hard-core parameter ε in logarithmic scale, for $k = 3$ and $l = 1/2$, in the completely isotropic case ($\varepsilon_{0,0} = \varepsilon_{1,m} = \varepsilon$, for $m = 1, 0, -1$).

Chapter 5

Statistical Potential for NACS Particles

5.1 Introduction

A key property of the statistics of quantum systems in two space dimensions is provided by the possibility to display intermediate fractional statistics interpolating between bosons and fermions: the properties of anyons, the two-dimensional identical particles obeying fractional braiding statistics and carrying fractional charge, have been the subject of an intense and continuing interest [14, 10, 7, 73, 74]. Both Abelian and non-Abelian anyons (associated to respectively one-dimensional and irreducible higher-dimensional representations of the braid group) have been extensively studied both for their intrinsic interest and their connection with quantum Hall systems [74, 79, 81]. In particular there is an increasing interest in the investigation of the properties of non-Abelian anyons for their application to topologically fault-tolerant quantum information processing [74].

The ideal gas of anyons is also interesting for the phenomenon of statistical transmutation [10, 7, 73], i.e. the fact that one can treat non-interacting anyons as interacting bosons or fermions. The idea behind statistical transmutation is that one can alternatively consider the system as made of interacting particles with canonical statistics or of non-interacting particles, but obeying non-canonical statistics. This makes in general difficult the study of the equilibrium thermodynamical properties of the ideal anyon gases [73] and therefore any result that gives even qualitative informations on such ideal gases is valuable.

Dating back to the seminal work by Uhlenbeck and Gropper [129], a standard way to characterize the effects of the quantum statistics on the properties of ideal gases is provided by the determination of the so-called statistical potential. As detailed in textbooks [130, 114], one can show that the partition function (PF) of a gas of particles approaches - for sufficiently high temperatures - the PF of the classical gas (with the correct Boltzmann counting). When this computation is done for an ideal quantum gas (under the condition that the thermal wavelength is much smaller than the interparticle distance) one finds that the quantum PF becomes the PF of the classical ideal gas. Evaluating the first quantum correction, one can appreciate that the quantum PF can be formally written as the PF a classical gas in which a fictitious two-body interaction term (the statistical potential) is added [130, 114]. The statistical potential gives a simple characterization of the effects of the quantum statistics of the ideal gases: the statistical potential is "attractive" for bosons and "repulsive" for fermions, and it is respectively monotonically increasing (decreasing) for bosons (fermions). Another important result is that a suitable integral of the statistical interparticle potential gives the second coefficient of the virial expansion [130, 114].

The statistical potential of a gas of ideal Abelian anyons has been studied in [131, 132, 133] and it depends on the statistical parameter α (we remind that $\alpha = 0$ and $\alpha = 1$ corresponds respectively to free two-dimensional spinless bosons and fermions, while $\alpha = 1/2$ corresponds to semions [73]). It is found that for $1/2 \leq \alpha \leq 1$ the statistical potential $v_\alpha(r)$ is monotonically decreasing, while for $0 < \alpha < 1/2$ it has a a minimum at a finite value of r and it is increasing for larger values of r (while for $\alpha = 0$ is monotonically increasing). We can refer to these behaviours respectively as "quasi-fermionic" ($1/2 \leq \alpha < 1$) and "quasi-bosonic" ($0 < \alpha < 1/2$). The purpose of the present Chapter is to compute the statistical potential of the ideal gas of non-Abelian anyons: we find that the behaviour of the statistical potential depends on the Chern-Simons coupling and the isospin quantum number. As a function of these two parameters, quasi-bosonic and quasi-fermionic regions emerge, and they are part of a phase diagram which will be presented below. Furthermore, a third behaviour ("bosonic-like") appears in this phase diagram, corresponding to a monotonically increasing statistical potential.

The plan of the Chapter is the following: in Section 2 we recall the steps which lead to the computation of the statistics potential v_α of an ideal gas of Abelian anyons with statistical parameter α. Since it is possible to show that the statistical potential for the non-Abelian gas may be written in terms of sums of Abelian statistical potentials (having different statistical parameters depending on the projection of the isospin quantum number), we provide in Section 2 a detailed study of v_α: we present a compact and useful integral representation for it, showing that it can be written in terms of bivariate Lommel functions [134]. Furthermore, using the Sumudu transform of the statistical potential [135], we also give a closed expression of $v_\alpha(r)$ in terms of the inverse Laplace transform of an algebraic function of r and α. Using this result we are able to give a simple expression for the statistical potential of the ideal gas of semions, and we show that for a general value α it is possible to retrieve the well-known result for the second virial coefficient of an ideal anyon gas found in [91] (with an hard-core boundary condition for the two-body wavefunction at zero distance). For the sake of comparison with the non-Abelian case that follows, the limit behaviours both at small and large distance of the statistics potentials are presented. In Section 3 we introduce the non-Abelian Chern-Simons (NACS) model studied in the rest of the Chapter: we compute the statistics potential (within the hard-core boundary condition frame) as a function of the Chern-Simons coupling κ and the isospin quantum number l and we build a phase diagram summarizing the behaviour of the statistical potential in terms of κ and l. We then show that the second virial coefficient, previously studied in [85, 110, 111, 2], is correctly retrieved. The asymptotic expressions for the small and large distance of the statistics potentials are also given.

5.2 Statistical Potential for Abelian Anyons

In this Section we introduce the model for an ideal gas of Abelian anyons, and we then derive its statistical potential $v_\alpha(r)$ as a function of the statistical parameter α, obtaining the expression for $v_\alpha(r)$ given in [131, 132, 133], and also providing an explicit formula for the semions (half-integer values of α). The results for $v_\alpha(r)$ and the asymptotic expressions for small and large distance will be used in the next Section, where the statistical inter-particle potential of an ideal gas of non-Abelian anyons is derived and studied.

Abelian anyons admit a concrete representation by the flux-charge composite model [73], and the statistics of these objects can be understood in terms of Aharonov-Bohm type interference [115, 136]. The Hamiltonian for the quantum dynamics of an ideal system of anyons reads [7, 73]

$$H_N = \sum_{n=1}^{N} \frac{1}{2M} \left(\vec{p}_n - \alpha\, \vec{a}_n\right)^2 \ , \tag{5.1}$$

where $\vec{p}_n$ is the momentum of the n-th particle ($n = 1, \cdots, N$). Similarly we will denote the position of the n-th particle by $\vec{r}_n \equiv (x_n^1, x_n^2)$. In Eq.(5.1) α is the statistical parameter: notice that the physical quantities, e.g. the virial coefficients, are periodic with period 2 [73]: the bosonic points are defined by $\alpha = 2j$ and the fermionic ones by $\alpha = 2j + 1$, j integer. For this reason we will consider in the following $\alpha \in [0, 2]$.

In Eq.(5.1) $\vec{a}_n$ is the vector potential carrying the flux attached to the bosons: indeed, the Hamiltonian (5.1) is written in the so-called bosonic representation, i.e. it is an Hamiltonian for identical bosons, and therefore acting on the subspace of wavefunctions which are symmetric with respect to the exchange of particles. The explicit expression for $\vec{a}_n \equiv (a_n^1, a_n^2)$ is

$$a_n^i = \hbar\, \epsilon^{ij} \sum_{m(\neq n)} \frac{x_n^j - x_m^j}{|\vec{r}_n - \vec{r}_m|^2} \ . \tag{5.2}$$

where ϵ^{ij} is the totally antisymmetric tensor $(i, j = 1, 2)$.

Let's recall that this model of Abelian anyons also admits a field-theoretic description: in fact (non-relativistic) anyons can be described by bosonic Schrödinger fields ψ and $\psi^\dagger$ coupled to a Chern-Simons gauge field a_μ living in (2+1)-D [118, 119] (then $\mu = 0, 1, 2$). The Lagrangian density of such a system reads

$$\mathcal{L} = \frac{c}{2}\, \epsilon^{\mu\nu\lambda}\, a_\mu \partial_\nu a_\lambda + \psi^\dagger \left(i\, D_t + \frac{1}{2M} \vec{D}^2 \right) \psi$$

where c gives the measure of the interaction among particles mediated by the $U(1)$ gauge potential a_μ, with the covariant derivatives given by $D_t = \partial_t + iqa_0$, $\vec{D} = \vec{\nabla} - iq\vec{a}$, and the anyon statistical parameter α to be identified as $\alpha \equiv q^2/(2\pi c)$.

In the study of a quantum-mechanical ideal gas, the effect of the symmetry properties of the wave function can be interpreted, from a classical point of view, as the consequence of a fictitious classical potential introduced by Uhlenbeck and Gropper [129], referred to as effective statistical potential, which represents the first quantum correction for the classical PF [130, 114]. For our purposes we have to consider the two-body case, which is relevant for the subsequent computation of the statistical interparticle potential [130, 114]. The statistical potential completely determines the second virial coefficient, which gives the thermodynamical properties of the system in the dilute (high-temperature) regime. For the two-anyon system, after separating in (5.1) with $N = 2$ the center-of-mass dynamics (i.e. that of a particle having mass $2M$), one is left with the dynamics of the relative wavefunction:

$$-\frac{1}{M} \left[\vec{\nabla}_r - i\, \alpha\, \vec{a}(\vec{r}) \right]^2 \psi(\vec{r}) = E\, \psi(\vec{r}) \ , \tag{5.3}$$

where $\vec{r} = \vec{r}_1 - \vec{r}_2$ is the relative coordinate, and $a^i(\vec{r}) = \epsilon^{ij} \frac{x^j}{r^2}$. Therefore the relative dynamics is equivalent to that of a single particle in presence of a point vortex $\vec{a}(\vec{r})$ placed at the origin.

An analysis of the statistical interparticle potential for an ideal gas of Abelian anyons is presented in [131, 132]. The eigenfunctions of (5.3) read

$$\Psi_\alpha = e^{i\mathbf{K}\cdot\mathbf{R}}\, e^{il\theta}\, J_{|l-\alpha|}(kr) \equiv e^{i\mathbf{K}\cdot\mathbf{R}}\, \psi_\alpha \ , \tag{5.4}$$

where the capital (italic) letters respectively refer to center-of-mass (relative) coordinates. The bosonic description used in (5.1) imposes the condition $l =$ even; furthermore $J_m(x)$ denote the Bessel functions of the first kind [134] (their definition is recalled in Appendix 5.3.2). Notice that the wavefunctions (5.4) are the eigenfunctions of (5.3) provided that the hard-core boundary conditions $\Psi_\alpha(0) = 0$ are imposed.

The two-body PF is $Z = \mathrm{Tr}\, e^{-\beta H_2} = 2A\lambda_T^{-2} Z'$, where Z' is the single-particle PF in the relative coordinates, $\beta = 1/k_B T$, $\lambda_T = (\beta h^2/2\pi M)^{1/2}$ is the thermal wavelength and A is the area of the system. The relative PF Z' is given by

$$Z' = \frac{1}{h^2} \sum_{l=-\infty}^{\infty} \int d^2p \int d^2r\ e^{-\beta p^2/M}\, |\psi_\alpha|^2 \tag{5.5}$$

(with $p = \hbar k$). It is possible to conveniently rewrite Eq.(5.5) by using the following integral property [137] of the Bessel functions:

$$\int_0^\infty e^{-\alpha x} J_\nu \left(2\beta\sqrt{x}\right)\, J_\nu \left(2\gamma\sqrt{x}\right) dx = \frac{1}{\alpha}\, I_\nu \left(\frac{2\beta\gamma}{\alpha}\right)\ e^{-(\beta^2+\gamma^2)/\alpha} \ , \tag{5.6}$$

where I_ν is the modified Bessel function of the first kind [134] (see also Appendix 5.3.2). The relative PF then takes the form

$$Z' = \frac{1}{2} \sum_{n=-\infty}^{\infty} \int_0^\infty dx\, e^{-x}\, I_{|2n-\alpha|}(x) \ , \tag{5.7}$$

where

$$x \equiv \frac{Mr^2}{2\beta\hbar^2} = \frac{\pi r^2}{\lambda_T^2} \ . \tag{5.8}$$

The one-body PF for classical systems, used in [129] to define the concept of effective statistical potential, is

$$Z' = \frac{1}{2h^2} \int d^2p\, e^{-\beta p^2/M} \int d^2r\, e^{-\beta v(r)} \ : \tag{5.9}$$

therefore, comparing (5.7) and (5.9) produces as a result [131]

$$e^{-\beta v_\alpha(r)} = 2\, e^{-x} \sum_{n=-\infty}^{\infty} I_{|2n-\alpha|}(x) \tag{5.10}$$

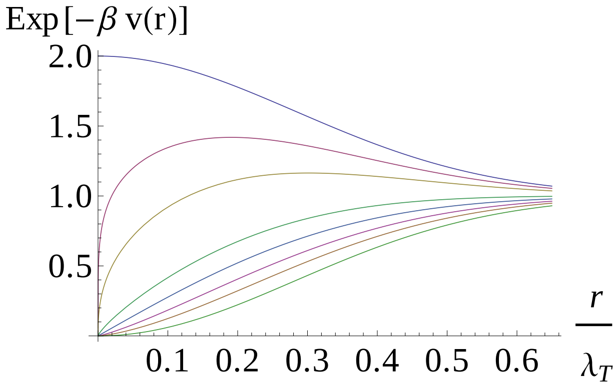

Figure 5.1: Plot of $e^{-\beta v_\alpha(r)}$ vs. r/λ_T for different values of the statistical parameter: from top to the bottom it is $\alpha = 0, 0.1, 0.2, 0.4, 0.5, 0.6, 0.7, 1$.

(see Appendix 5.3.2 for details). Eq.(5.10) is plotted in Fig.5.1.

For the considerations which follow, it is convenient to introduce the function

$$\mathcal{M}_\alpha(x) \equiv \sum_{n=-\infty}^{\infty} I_{|2n-\alpha|}(x) \ , \tag{5.11}$$

so that the statistical potential can be written as

$$e^{-\beta v_\alpha(r)} = 2\, e^{-x}\, \mathcal{M}_\alpha(x) \ . \tag{5.12}$$

The inter-particle statistical potential admits a closed expression in terms of the bivariate Lommel functions (alias, Lommel functions of two variables). Indeed, as evident from Appendix 5.3.2, it is

$$\mathcal{M}_\alpha(x) = i^{-\alpha}\, U_\alpha(ix, ix) - i^{\alpha}\, U_{2-\alpha}(ix, ix) \ , \tag{5.13}$$

where U_α denote the Lommel functions of two variables [134]. Notice that the symmetry property $v_\alpha(r) = v_{2-\alpha}(r)$ $(\forall \alpha \in \mathcal{R}\ )$ holds for the statistical potential.

In Appendix 5.3.2 we prove the following integral representation for $v_\alpha(r)$, which will result useful in Subsection 5.2.1 for discussing the large-distance limit behaviour:

$$e^{-\beta v_\alpha(r)} = 1 + e^{-2x} \cos\alpha\pi - 2\, \frac{\sin\alpha\pi}{\pi}\, e^{-x} \int_0^\infty dt\, \frac{\sinh\left[(1-\alpha)t\right]}{\sinh t}\, e^{-x\cosh t} \ . \tag{5.14}$$

Other integral representations for $e^{-\beta v_\alpha(r)}$ are presented in Appendix 5.3.2. In the remaining parts of this Section we discuss the asymptotic behaviour of

v_α for small and large values of the dimensionless parameter x, and we give a closed expression for the statistical potential $v_\alpha(r)$ in terms of the inverse Laplace transform of an algebraic function of r and α: this manipulation allows us to straightforwardly regain the known expressions for $v_\alpha(r)$ in the bosonic/fermionic cases, to obtain its expression in the case of semions and to finally recover the value of the second virial coefficient for a generic α presented in [91].

5.2.1 Limit behaviours

In order to quantitatively understand the tendency of anyons to bunch together or vice versa to repel each other in different limit regimes of density, let's recall and further discuss the asymptotic behaviours of their effective statistical potential, for small and large distances [132, 133].

For small r (that is $x \ll 1$) we can approximate the summation term in (5.10) as

$$\sum_{n=-\infty}^{\infty} I_{|2n-\alpha|}(x) \approx \sum_{n=0}^{\infty}\left[\frac{1}{\Gamma(2n+\alpha+1)}(x/2)^{2n+\alpha} + \frac{1}{\Gamma(2n+3-\alpha)}(x/2)^{2n+2-\alpha}\right] \approx$$
$$\approx [\Gamma(\alpha+1)]^{-1}(x/2)^{\alpha} + [\Gamma(3-\alpha)]^{-1}(x/2)^{2-\alpha} \ . \tag{5.15}$$

Since

$$\beta v_s(r) = -\ln\left[2\,e^{-x}\sum_{n=-\infty}^{\infty} I_{|2n-\alpha|}(x)\right] \approx -\ln\left[2\,e^{-x}\left([\Gamma(\alpha+1)]^{-1}(x/2)^{\alpha} + [\Gamma(3-\alpha)]^{-1}(x/2)^{2-\alpha}\right)\right] \ , \tag{5.16}$$

it follows

$$\beta v_\alpha(r) \approx \begin{cases} -\ln\left[2 - 2\pi r^2/\lambda_T^2\right] \approx -\ln 2 + \frac{\pi}{2}(r/\lambda_T)^2, & \alpha = 0,\ 2 \\ -\ln\left[2\,(\pi r^2/2\,\lambda_T^2)^{\alpha}/\Gamma(\alpha+1)\right], & 0<\alpha<1 \\ -\ln\left[2\pi r^2/\lambda_T^2\right], & \alpha = 1 \\ -\ln\left[2\,(\pi r^2/2\,\lambda_T^2)^{2-\alpha}/\Gamma(3-\alpha)\right], & 1<\alpha<2 \end{cases} \tag{5.17}$$

We may summarize the small distance behaviour as follows: $v_\alpha(r)$ is repulsive and logarithmically divergent to $+\infty$ for any $\alpha \in (0,2)$, whereas for $\alpha = 0, 2$ it is attractive, and quadratically increasing in r starting from the finite value $v_0(r=0) = -\ln 2$. The small-r asymptotic function for $v_\alpha(r)$ is discontinuous in $\alpha = 1$, being twice than the limit asymptotic functions for $\alpha \to 1^{\pm}$ (in fact two equal Bessel terms contribute to the asymptotic behaviour for $\alpha = 1$, whereas only one of them dominates when $\alpha \neq 1$).

To study the behaviour of the statistical potential for large distance r (i.e. $x \gg 1$) we employ the integral representation (5.14). The method of steepest descent allows to evaluate to an arbitrary order the last term of this integral representation. At the first significant order, we get

$$-2\,\frac{\sin\alpha\pi}{\pi}\,e^{-z}\int_0^\infty dt\,\frac{\sinh\left[(1-\alpha)t\right]}{\sinh t}\,e^{-z\,\cosh t}\approx\frac{2(\alpha-1)\sin\alpha\pi}{\sqrt{2\pi z}}e^{-2z}\,. \tag{5.18}$$

Therefore the large distance behaviour of the statistical potential is given by

$$\beta v_s(r)\approx\left[-\cos\alpha\pi+\frac{\sqrt{2}(1-\alpha)\sin\alpha\pi}{\pi r/\lambda_T}\right]\,e^{-2\pi r^2/\lambda_T^2} \tag{5.19}$$

for any $\alpha\in[0,2]$. Let us notice that this result differs from the corresponding one in [132], and that the asymptotic behaviours for $\alpha=0,\frac{1}{2},1$ (see formulas (5.26), (5.27), (5.28) in the sequel) are correctly retrieved. The statistical potential for large distance is vanishing for $r\to\infty$, and the interval $\alpha\in[0,1]$ (as well as the interval $\alpha\in[1,2]$, due to the symmetry property $v_\alpha=v_{2-\alpha}$) is divided in two regions: for large distance, $v_\alpha(r)$ is attractive for $0\leq\alpha<1/2$, and repulsive for $1/2\leq\alpha\leq1$.

The large-distance and short-distance behaviours, considered together, imply that $v_\alpha(r)$ must admit a minimum point at finite distance, $r_{cr}(\alpha)$ for any $0<\alpha<1/2$ (see Fig.5.2). We denote the corresponding dimensionless quantity by $x_{cr}(\alpha)\equiv\pi r_{cr}^2(\alpha)/\lambda_T^2$. The minimum point $x_{cr}(\alpha)$ tends to $+\infty$ for $\alpha\to\frac{1}{2}^-$, as shown in Fig.5.3.

Fig.5.2 clearly shows that Abelian anyons have, from the point of view of the statistical potential, a "quasi-bosonic" behaviour for $0<\alpha<1/2$ (i.e., $v_\alpha(r)$ is non-monotonic with a minimum) and a "quasi-fermionic" behaviour for $1/2\leq\alpha<1$ ($v_\alpha(r)$ is monotonically decreasing without a minimum). Obviously, for $\alpha=0$ there is a "bosonic-like" behaviour, and $v_\alpha(r)$ is monotonically increasing.

5.2.2 Laplace transform of the statistical potential and the 2^{nd}-virial coefficient

In this Section we write an explicit formula for $e^{-\beta v_\alpha(x)}$ as the inverse Laplace transform of a function of x and α. This result will allow us to write a simple formula for the statistical potential for the semionic gas and to compute the second virial coefficient, which of course coincides with the result reported in the seminal reference [91].

Let's start by writing down the Sumudu transform [135] of the function

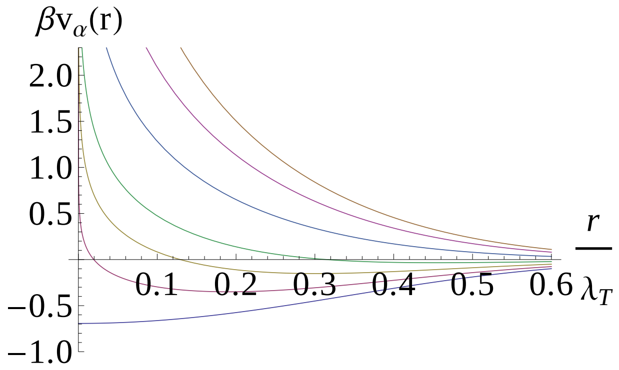

Figure 5.2: $\beta v_\alpha(r)$ vs. r/λ_T for different values of α: from top to bottom it is $\alpha = 1, 0.7, 0.5, 0.3, 0.2, 0.1, 0$. The bosonic curve ($\alpha = 0$) is monotonically increasing while the fermionic curve ($\alpha = 1$) is monotonically decreasing and divergent for $r \to 0$. All the curves for $0 < \alpha < 1/2$ diverge for $r \to 0$ and have a minimum point at finite r.

$\mathcal{M}_\alpha$, defined in Eq.(5.11). The Sumudu transform of $\mathcal{M}_\alpha$ is defined as

$$[\mathcal{M}_\alpha(x)]_S \equiv \int_0^\infty e^{-\theta}\mathcal{M}_\alpha(\theta x)d\theta = \int_0^\infty e^{-t/x}\mathcal{M}_\alpha(t)\frac{dt}{x} = s\int_0^\infty e^{-st}\mathcal{M}_\alpha(t)dt \ , \tag{5.20}$$

where $s \equiv \frac{1}{x}$, whence

$$\left[\mathcal{M}_\alpha\left(\frac{1}{x}\right)\right]_S = x\,\mathcal{L}[\mathcal{M}_\alpha](x) \ , \tag{5.21}$$

$\mathcal{L}$ being the ordinary (one-sided) Laplace transform. The function $\mathcal{M}_\alpha(x)$ is given by

$$\mathcal{M}_\alpha(x) = \sum_{n=-\infty}^{\infty} I_{|2n-\alpha|}(x) = e^{-\gamma\pi/2i}\sum_{n=0}^{\infty}(-1)^n J_{2n+\gamma}(ix) + e^{-\alpha\pi/2i}\sum_{n=0}^{\infty}(-1)^n J_{2n+\alpha}(ix) \tag{5.22}$$

where $\gamma \equiv 2 - \alpha$. Substituting (5.22) in (5.20), one gets that the Sumudu transform of $\mathcal{M}_\alpha$ becomes

$$[\mathcal{M}_\alpha(x)]_S = e^{-\gamma\frac{\pi}{2}i}\sum_{n=0}^{\infty}(-1)^n\int_0^\infty e^{-t}J_{2n+\gamma}(ixt)dt + e^{-\alpha\frac{\pi}{2}i}\sum_{n=0}^{\infty}(-1)^n\int_0^\infty e^{-t}J_{2n+\alpha}(ixt)dt \ .$$

The use of the integral properties of the Bessel functions of the first kind [134] (see pg. 386) gives the following expression:

$$[\mathcal{M}_\alpha(x)]_S = \frac{e^{-\gamma\frac{\pi}{2}i}}{\sqrt{1-x^2}}\sum_{n=0}^{\infty}(-1)^n\left(\frac{\sqrt{1-x^2}-1}{ix}\right)^{2n+\gamma} +$$

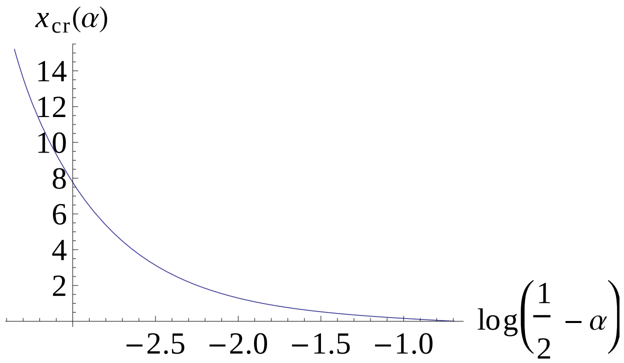

Figure 5.3: The dimensionless value of the minimum point $x_{cr}(\alpha)$ of the statistical potential v_α of Abelian anyons as a function of the statistical parameter α near $\alpha = 1/2$.

$$+\frac{e^{-\alpha\frac{\pi}{2}i}}{\sqrt{1-x^2}}\sum_{n=0}^{\infty}(-1)^n\left(\frac{\sqrt{1-x^2}-1}{ix}\right)^{2n+\alpha} =$$

$$= \frac{1}{\sqrt{1-x^2}}\left\{\left(\frac{(1-\sqrt{1-x^2})}{x}\right)^{\gamma}+\left(\frac{(1-\sqrt{1-x^2})}{x}\right)^{\alpha}\right\}\frac{1}{1-\left(\frac{1-\sqrt{1-x^2}}{x}\right)^2} =$$

$$= \frac{1}{2}\left(\frac{1}{1-x^2}+\frac{1}{\sqrt{1-x^2}}\right)\left[\left(\frac{(1-\sqrt{1-x^2})}{x}\right)^{\gamma}+\left(\frac{(1-\sqrt{1-x^2})}{x}\right)^{\alpha}\right]. \tag{5.23}$$

By sending $x \to 1/x$ in (5.23), and applying (5.21), we obtain

$$\mathcal{M}_\alpha\left(\frac{1}{x}\right)_S = \frac{1}{2}\left(\frac{x^2}{x^2-1}+\frac{x}{\sqrt{x^2-1}}\right)\left[\left(x-\sqrt{x^2-1}\right)^{\gamma}+\left(x-\sqrt{x^2-1}\right)^{\alpha}\right]$$

and

$$\mathcal{L}[\mathcal{M}_\alpha](x) = \frac{\left(x-\sqrt{x^2-1}\right)^{1-\alpha}+\left(x-\sqrt{x^2-1}\right)^{\alpha-1}}{2(x^2-1)}. \tag{5.24}$$

Hence the inter-particle statistical potential admits the following form, for many purposes easier to handle than (5.10), since it does not contain infinite sums:

$$e^{-\beta v_\alpha(r)} = e^{-x}\mathcal{L}^{-1}\left[\frac{\left(x-\sqrt{x^2-1}\right)^{1-\alpha}+\left(x-\sqrt{x^2-1}\right)^{\alpha-1}}{x^2-1}\right]. \tag{5.25}$$

The correct potentials for the bosonic and fermionic cases are straightforwardly reproduced: using known results for the Laplace transforms [138], we get

$$e^{-\beta v_{\alpha=0}(r)} = e^{-x}\mathcal{L}^{-1}\left[\frac{2x}{x^2-1}\right] = e^{-x}\,2\cosh x = 1+e^{-2x}\ ; \tag{5.26}$$

$$e^{-\beta v_{\alpha=1}(r)} = e^{-x}\mathcal{L}^{-1}\left[\frac{2}{x^2-1}\right] = e^{-x}\,2\sinh x = 1-e^{-2x}\ . \tag{5.27}$$

Eq.(5.25) gives a closed formula for the potential in the case of semions ($\alpha = 1/2$, or $\alpha = 3/2$):

$$e^{-\beta v_{\mathrm{sem.}}(r)} = e^{-x}\mathcal{L}^{-1}\left[\frac{\left(x-\sqrt{x^2-1}\right)^{1/2}+\left(x-\sqrt{x^2-1}\right)^{-1/2}}{x^2-1}\right] = \mathrm{erf}(\sqrt{2x}) \tag{5.28}$$

where erf is the error function [139].

Eq.(5.25) allows us to recover the second virial coefficient of a gas made of identical Abelian $\alpha-$anyons, which is given by [91]:

$$B_2(\alpha,T) = \frac{1}{4}\lambda_T^2\left(-1+4\alpha-2\alpha^2\right)\ . \tag{5.29}$$

The link between the 2^{nd}-virial coefficient and the statistical potential can be expressed in the form

$$B_2(\alpha,T) = \frac{\lambda_T^2}{2}\int_0^\infty dx\,[1-e^{-\beta v(x)}]\ . \tag{5.30}$$

Using Eqs.(5.26)-(5.27)-(5.28), for the three special cases $\alpha = 0$, 1 and 1/2 (corresponding respectively to bosons, fermions and semions) one immediately finds

$$B_2(\alpha=0,T) = \frac{\lambda_T^2}{2}\int_0^\infty dx\,[1-(1+e^{-2x})] = -\frac{1}{4}\lambda_T^2\ ; \tag{5.31}$$

$$B_2(\alpha=1,T) = \frac{\lambda_T^2}{2}\int_0^\infty dx\,[1-(1-e^{-2x})] = +\frac{1}{4}\lambda_T^2\ ; \tag{5.32}$$

$$B_2(\alpha=\frac{1}{2},T) = \frac{\lambda_T^2}{2}\int_0^\infty dx\,[1-\mathrm{erf}\,(\sqrt{2x})] = \frac{\lambda_T^2}{2}\int_0^\infty dy\,y\,[1-\mathrm{erf}\,y] = \frac{1}{8}\lambda_T^2 \tag{5.33}$$

as it should be.

Finally, the effective 2-body statistical potential written as in Eq. (5.25) allows us to easily recover the expression of the 2^{nd}-virial coefficient even for a general statistical parameter α. Indeed, by virtue of (5.25) and the dominated convergence theorem, one has

$$\frac{B_2(\alpha,T)}{\lambda_T^2} = \frac{1}{2}\int_0^\infty dx\,[1-e^{-\beta v(x)}]$$
$$= \frac{1}{2}\lim_{\varepsilon\to 0}\int_0^\infty dx\left[e^{-\varepsilon x} - e^{-(1+\varepsilon)x}\left(\mathcal{L}^{-1}\left[\frac{\left(x-\sqrt{x^2-1}\right)^{1-\alpha}}{x^2-1} - \frac{\left(x-\sqrt{x^2-1}\right)^{\alpha-1}}{x^2-1}\right]\right)\right]$$
$$= \frac{1}{2}\lim_{\varepsilon\to 0}\left[\frac{1}{\varepsilon} - \mathcal{L}_{|x=1+\varepsilon}\left(\mathcal{L}^{-1}\left[\frac{\left(x-\sqrt{x^2-1}\right)^{1-\alpha}}{x^2-1}\right] - \mathcal{L}^{-1}\left[\frac{\left(x-\sqrt{x^2-1}\right)^{\alpha-1}}{x^2-1}\right]\right)\right]$$
$$= \frac{1}{2}\lim_{\varepsilon\to 0}\left[\frac{1}{\varepsilon} - \frac{(1+\varepsilon-\sqrt{2\varepsilon+\varepsilon^2})^{1-\alpha} + (1+\varepsilon-\sqrt{2\varepsilon+\varepsilon^2})^{\alpha-1}}{2\varepsilon+\varepsilon^2}\right] = \frac{1}{4}(-1+4\alpha-2\alpha^2)\ , \tag{5.34}$$

that is just (5.29).

As a byproduct of Eqs. (5.12), (5.13), (5.30), (5.34), we find an interesting integral property relevant to the bivariate Lommel functions (new, at the best of our knowledge):

$$\int_0^\infty dx\,\left\{1-2\,e^{-x}\left[i^{-\alpha}\,U_\alpha(ix,ix) - i^\alpha\,U_{2-\alpha}(ix,ix)\right]\right\} = -\frac{1}{2}+2\alpha-\alpha^2\ . \tag{5.35}$$

5.3 Statistical Potential For Non-Abelian Anyons

In this Section we discuss the statistical interparticle potential for a two-dimensional system of $SU(2)$ NACS spinless particles. The NACS particles are pointlike sources mutually interacting via a non-Abelian gauge field attached to them [89]. As a consequence of their interaction, equivalent to a non-Abelian statistical interaction for a system of bosons, they are endowed with fractional spins and obey braid statistics as non-Abelian anyons.

Let's briefly introduce the NACS quantum mechanics [122, 38, 123, 104]. The Hamiltonian describing the dynamics of the N-body system of free NACS particles can be derived by a Lagrangian with a Chern-Simons term and a matter field coupled with the Chern-Simons gauge term [104]: the resulting Hamiltonian reads

$$H_N = -\sum_{i=1}^N \frac{1}{M_i}\left(\nabla_{\bar{z}_i}\nabla_{z_i} + \nabla_{z_i}\nabla_{\bar{z}_i}\right) \tag{5.36}$$

where M_i is the mass of the i-th particles, $\nabla_{\bar{z}_i} = \frac{\partial}{\partial \bar{z}_i}$ and

$$\nabla_{z_i} = \frac{\partial}{\partial z_i} + \frac{1}{2\pi\kappa} \sum_{j \neq i} \hat{Q}_i^a \hat{Q}_j^a \frac{1}{z_i - z_j} \ .$$

In formula (4.15) $i = 1, \ldots, N$ labels the particles, $(x_i, y_i) = (z_i + \bar{z}_i, -i(z_i - \bar{z}_i))/2$ are their spatial coordinates, and $\hat{Q}^a$'s are the isovector operators which can be represented by some generators T_l^a in a representation of isospin l [123]. The quantum number l labels the irreducible representations of the group of the rotations induced by the coupling of the NACS particle matter field with the non-Abelian gauge field: as a consequence, the values of l are of course quantized and vary over all the integer and the half-integer numbers, with $l = 1/2$ being the smallest possible non-trivial value ($l = 0$ corresponds to a system of free bosons). As usual, a basis of isospin eigenstates can be labeled by l and the magnetic quantum number m (varying in the range $-l, -l+1, \cdots, l-1, l$).

Hence the statistical potential depends in general on the value of the isospin quantum number l and on the coupling κ (and of course on the distance r and the temperature T). The quantity κ present in the covariant derivative is a parameter of the theory. The condition $4\pi\kappa =$ integer has to be satisfied for consistency reasons [44, 122]. In the following we denote for simplicity by k the integer $4\pi\kappa$.

For non-Abelian anyons, in analogy with (5.9), the effective statistical potential can be related to the relative PF according to the following expression:

$$Z_2'(\kappa, l, T) - Z_2'^{(n.i.)}(l, T) = \frac{1}{2h^2} \int d^2p \, e^{-\beta p^2/M} \int d^2r \left[\exp[-\beta v(\kappa, l, r)] - \exp[-\beta v^{(n.i.)}(l, r)] \right] \ , \tag{5.37}$$

where $v^{(n.i.)}(l, r)$ refers to the system of particles with isospin l and without statistical interaction ($\kappa \to \infty$). The potential $v^{(n.i.)}(l, r)$ can be expressed in terms of the potentials $v_{\alpha=0}(r)$ and $v_{\alpha=1}(r)$ for the free Bose and Fermi systems (endowed with the chosen wave-function boundary conditions). $Z_2'(\kappa, l, T) - Z_2'^{(n.i.)}(l, T)$ is the (convergent) variation of the divergent PF for the two-body relative Hamiltonian, between the interacting case in exam and the non-interacting limit $\kappa \to \infty$.

For hard-core boundary conditions on the relative two-anyonic vectorial wave-function, the quantity $v^{(n.i.)}(l, r)$ which enters Eq.(5.37) is given by the projection onto the bosonic/fermionic basis:

$$e^{-\beta v^{(n.i.)}(l,r)} = \frac{1}{(2l+1)^2}\sum_{j=0}^{2l}(2j{+}1)\left[\frac{1+(-1)^{j+2l}}{2}\,e^{-\beta v_{\alpha=0}(r)}+\frac{1-(-1)^{j+2l}}{2}\,e^{-\beta v_{\alpha=1}(r)}\right]\ , \tag{5.38}$$

in analogy with the procedure shown in [111, 110, 2] for the computation of the 2^{nd}-virial coefficient. By using the results $e^{-\beta v_{\alpha=0}(r)} = 1 + e^{-2x}$, $e^{-\beta v_{\alpha=1}(r)} = 1 - e^{-2x}$, one then obtains

$$\exp[-\beta v^{(n.i.)}(l,r)] = 1 + \frac{e^{-2x}}{2l+1}\ . \tag{5.39}$$

Notice that this non-interacting quantity exactly corresponds to $(-1)^{2l}$ times the statistical potential for a system of identical $(2l)$-spin ordinary particles (fulfilling the spin-statistics constraint) at the same temperature, similarly to what argued in [2] about the 2^{nd}-virial coefficient for the same system.

In the interacting case (i.e. finite k), we can express the statistical potential in terms of statistical potentials of Abelian anyons:

$$e^{-\beta v(k,l,r)} = \frac{1}{(2l+1)^2}\sum_{j=0}^{2l}(2j{+}1)\left[\frac{1+(-1)^{j+2l}}{2}\,e^{-\beta v^B_{\omega_j}(r)}+\frac{1-(-1)^{j+2l}}{2}\,e^{-\beta v^F_{\omega_j}(r)}\right]\ , \tag{5.40}$$

where $\omega_j \equiv [j(j+1) - 2l(l+1)]/k$, and $v^B_{\omega_j}(r)$, $v^F_{\omega_j}(r)$ are the potentials for the Abelian ω_j-anyon gases respectively in the bosonic and fermionic bases, given by

$$e^{-\beta v^B_{\omega_j}(r)} = 2\,e^{-x}\mathcal{M}_{\omega_j}(x) \tag{5.41}$$

and

$$e^{-\beta v^F_{\omega_j}(r)} = 2\,e^{-x}\mathcal{M}_{\omega_j+1}(x)\ . \tag{5.42}$$

Both (5.41) and (5.42) are periodic quantities under the shift $\omega_j \to \omega_j + 2$; it follows

$$e^{-\beta v(k,l,r,T)} = \frac{2\,e^{-x}}{(2l+1)^2}\sum_{j=0}^{2l}(2j{+}1)\left[\frac{1+(-1)^{j+2l}}{2}\mathcal{M}_{\omega_j}(x)+\frac{1-(-1)^{j+2l}}{2}\mathcal{M}_{\omega_j+1}(x)\right]\ . \tag{5.43}$$

Eq.(5.43) gives the statistical potential for a gas of NACS particles.

5.3.1 2^{nd}-virial coefficient

An useful application (and check, at the same time) of Eq.(5.43) consists in computing the second virial coefficient. The analogous of (5.30) reads

$$B_2(k,l,T) = \frac{\lambda_T^2}{2}\int_0^\infty dx\,[1 - e^{-\beta v(k,l,r)}]\ . \tag{5.44}$$

Substituting in its integrand both Eq.(5.43) and the following decomposition of the unity

$$1 = \frac{1}{(2l+1)^2} \sum_{j=0}^{2l} (2j+1) \left[\frac{1+(-1)^{j+2l}}{2} + \frac{1-(-1)^{j+2l}}{2} \right] , \qquad (5.45)$$

one obtains for $B_2(k,l,T)$

$$\frac{\lambda_T^2}{2(2l+1)^2} \sum_{j=0}^{2l} (2j+1) \int_0^\infty dx \left[\frac{1+(-1)^{j+2l}}{2} \left(1 - 2e^{-x} \mathcal{M}_{\omega_j}(x)\right) + \frac{1-(-1)^{j+2l}}{2} \left(1 - 2e^{-x} \mathcal{M}_{\omega_j+1}(x)\right) \right] . \qquad (5.46)$$

By virtue of (5.35) one has then

$$B_2(k,l,T) = \frac{\lambda_T^2}{(2l+1)^2} \sum_{j=0}^{2l} (2j+1) \left[\frac{1+(-1)^{j+2l}}{2} \left(-\frac{1}{4} + \gamma_j - \frac{1}{2}\gamma_j^2 \right) + \right.$$
$$\left. + \frac{1-(-1)^{j+2l}}{2} \left(-\frac{1}{4} + \eta_j - \frac{1}{2}\eta_j^2 \right) \right]$$

where $\gamma_j \equiv \omega_j \, mod \, 2$ and $\eta_j \equiv (\omega_j + 1) \, mod \, 2$. This result matches with previous results reported in literature [111, 2], see in particular Eqs.(38) and (41) of [2].

5.3.2 Minimum points

Using Eq.(5.43), we can study if the gas of NACS has a "bosonic-like", "quasi-bosonic" or "quasi-fermionic" behaviour, according to its characterization in terms of the statistical potential. In correspondence with the analysis carried out in Subsection 5.2.1, we can address the problem of determining: which points of the discrete parameter space $\{k,l\}$ are associated to the presence of an (interior) minimum point $r_{crit}(k,l,T)$ for the statistical potential $v(k,l,r,T)$ (which will be referred to as "bosonic" ones), which points correspond to a monotonically increasing $v(k,l,r,T)$ ("bosonic-like" points) and which ones instead correspond to a $v(k,l,r,T)$ monotonically decreasing in r (the "quasi-fermionic" ones). To this aim, let's exploit the limit behaviours of $\exp[-\beta v(k,l,r,T)]$ for small distance ($x \ll 1$) and large distance ($x \gg 1$), which straightforwardly arise from Eqs.(5.15) and (5.18). Notice that at $x = 0$ one has

$$e^{-\beta v(k,l,r=0)} = \frac{1}{(2l+1)^2} \sum_{j=0}^{2l} (2j+1) \left[\left(1+(-1)^{j+2l}\right) \delta(\gamma_j, 0) + \left(1-(-1)^{j+2l}\right) \delta(\gamma_j, 1) \right] , \qquad (5.47)$$

where δ denotes the Kronecker delta function and $\gamma_j \equiv \omega_j mod\, 2$. For large distance it is

$$e^{-\beta v(k,l,r\gg\lambda_T)} \approx \begin{cases} 1+\frac{e^{-2x}}{(2l+1)^2}\, s(j,k,l), & \text{if } s(j,k,l)\neq 0 \\ 1-\frac{e^{-2x}}{(2l+1)^2\sqrt{2\pi x}}t(j,k,l), & \text{otherwise.} \end{cases} , \qquad (5.48)$$

where

$$s(j,k,l) \equiv \sum_{j=0}^{2l} (2j+1)(-1)^{j+2l} \cos(\omega_j \pi)$$

and

$$t(j,k,l) \equiv \sum_{j=0}^{2l} (2j{+}1)\sin(\gamma_j\pi)\left[(1+(-1)^{j+2l})(\gamma_j-1)-(1-(-1)^{j+2l})(\eta_j-1)\right]$$

with $\gamma_j \equiv \omega_j\, mod\, 2$, $\eta_j \equiv (\omega_j+1)\, mod\, 2$.

Our results can be summarized in the "phase diagram" shown in Fig.5.4, in which we distinguish pairs of parameters (k,l) for which $v(k,l,r,T)$ has a minimum point at finite r (in black), pairs for which the statistical potential is monotonically increasing in r (in magenta), and the remaining pairs (which are left blank), for which it is monotonically decreasing. In this way the black points denote a "quasi-bosonic" behaviour and the magenta ones denote a "bosonic-like" behaviour, according to the classification operated in Section II to extract information from the statistical potential for Abelian anyons. Bosonic-like behaviour occurs only for pairs (k,l) having one of the forms: $(k \text{ generic}, l=0)$, $(k=1, l \text{ integer})$, or $(k=2, l \text{ even})$. One sees that for non-Abelian anyons there are mixed regions in which quasi-bosonic and quasi-fermionic behaviour alternate, separated by regions dominated by a quasi-bosonic behaviour.

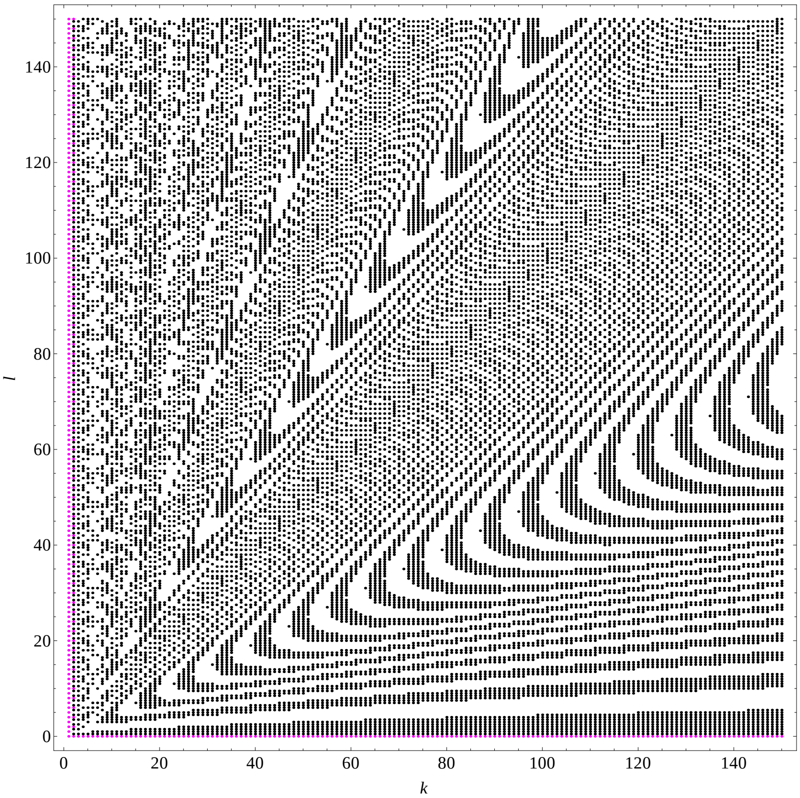

Figure 5.4: Phase diagram in the parameter space $\{k, l\}$ for the family of non-Abelian anyon models under consideration. Points (k, l) in the black regions correspond to the presence of an interior minimum point for the statistical potential $v(k, l, r, T)$ and are associated to quasi-bosonic behaviour. Points in magenta correspond to a bosonic-like, monotonically increasing, behaviour of the statistical potential. Finally, the remaining points are left blank and correspond to quasi-fermionic behaviour.

Conclusions

We have investigated some properties of Chern-Simons-based models, in which the Chern-Simons gauge field is considered in two respects. Firstly in its pure form of a Chern-Simons term, responsible alone for the whole action: in this case we have studied the concern of the Abelian (CS) surgery invariants with some homological features of the particles' closed worldlines. Secondly this gauge field is assumed being minimally coupled to a massive matter field, and our focus is then the thermodynamical description, made non-trivial because of the coupling itself.

About the first case (pure CS theory), by considering an even-dimensional tensorial reduced algebra T_{2k} of gauge-invariant charge states in the Abelian CS theory, we have studied the values of the link polynomial (associated to a non-trivial unknotted component C) defined in lens spaces, in dependence on the kind of lens spaces $L(p,1)$, on the coupling constant k and on the charge state I of the component C. Expectations for distinct lens spaces have been related each other thanks to a reciprocity lemma due to Cauchy and Kronecker.

Furthermore we have employed a rather refined kind of surgery presentation of homology spheres in order to prove a theorem of existence, whose meaning lies in identifying the expectations of links defined in homologically trivial manifolds with those of corresponding friend links living in the three-sphere. The Abelian specialization of the improved partition function associated to the CS theory has been offered, and has been shown that it is the unity for whatever homology sphere. A gauge-fixing procedure in (regular) three-manifolds is not available at the moment, so we cannot replace the surgery rules in the cases in which they are not applicable (as in some of the considered lens spaces). One could deepen the study of the Wilson loops values by means of a more serious number-theoretical approach, and make an attempt to cook a non-iterative algorithm for establishing when a Gauss sum vanishes, in reference to the applicability of the surgery rules. The discussion and the main theorem on the set of observables in homology three-spheres (and other statements described in due course) strongly insinuate a deep connection between the homology of a manifold and the behaviour of the Abelian CS theory in it defined beyond the cases of homologically trivial manifolds. More precisely, the verified indistinguishability of all three-homology spheres by the Abelian surgery invariant invites to study how the homological nature of a three-manifold affects its Abelian surgery invariant. A second application of the previously remembered reciprocity lemma has indicated the possibility of identifying p homological sectors in the space of the classes of fields defined on the manifold $L(p, 1)$, in such a way that the contribution of each sector (a fibre in the fiber bundle of the classes) to the partition function corresponds to a monomial term in the deformation parameter variable.

About the second model, in which the gauge field is coupled to matter, we studied the thermodynamical properties of an ideal gas of non-Abelian Chern-Simons particles considering the effect of general soft-core boundary conditions for the two-body wavefunction at zero distance. In comparison with the Abelian case, the thermodynamics of a system of free non-Abelian anyons appears to be much harder to study and all the available results were for hard-core boundary conditions, with - at the best of our knowledge - no results (even for the second virial coefficient) for soft-core non-Abelian anyons. The reason of this gap is at least twofold: from one side, for the difficulties, both analytical and numerical, in obtaining the finite temperature equation of state for non-Abelian anyons; from another side, because most of the efforts have been focused in the last decade on the study of two-dimensional systems which are gapped in the bulk and gapless on the edges, as for the states commonly studied for the fractional quantum Hall effect, while, on the contrary, the two-dimensional free gas of anyons is gapless. However, there is by now a mounting interest in the study of three-dimensional topological insulators, systems gapped in the bulk, but having protected conducting gapless states on their edge or surface: exotic states can occur at the surface of a three-dimensional topological insulator due to an induced energy gap, and a superconducting energy gap leads to a state supporting Majorana fermions, providing new possibilities for the realization of topological quantum computation. This surging of activity certainly calls for an investigation of the finite temperature properties of general gapless topological states on the two-dimensional surface of three-dimensional topological insulators and superconductors.

We determined and studied the second virial coefficient as a function of the coupling κ and the (iso)spin l for generic hard-core parameters. A discussion of the comparison of obtained findings with available results in literature for systems of non-Abelian hard-core Chern-Simons particles has been also supplied. We found that a semiclassical computation of the second virial coefficient for hard-core non-Abelian Chern-Simons particles gives the correct result, extending in this way the corresponding result for Abelian hard-core anyons. We have also written down the expressions for the thermodynamical quantities at the lowest order of the virial expansion, finding that at this order the relation between the internal energy and the pressure is the same found (exactly) for 2D Bose and Fermi ideal gases. Further studies on the higher virial coefficients are needed to establish the eventual validity of the obtained relation between the pressure and the internal energy for a general soft-core NACS ideal gas.

We have continued the investigation on this model by considering the two-body effective statistical potential (which models, in the dilute regime, the dominant term of the statistical interaction between the particles) of ideal systems of Abelian and non-Abelian anyons, described within the picture of ux-charge composites. In both cases we have derived closed expression for the statistical potential (in terms of known special functions) and we have studied its behavior. Asymptotic expansions have been provided, and the second virial coeffcients of both systems have been found using a compact expression of the statistical potential in terms of Laplace transforms. A phase diagram for the non-Abelian gas as a function of the Chern-Simons coupling and the isospin quantum number has been derived. In our study we have considered hard-core boundary conditions for the relative anyonic wave-functions: however, it would be interesting to use the results obtained so far in order to analyze the statistical potential in the more general soft-core conditions (both for the Abelian and non-Abelian ideal anyon gases).

Appendices

Computation of virial coefficients in special cases

To perform the comparison with Ref. [109], we compute in the following the virial coefficients for a NACS gas in the hard-core limit in the special cases considered in [109]:

$\star$ Case $l = 1/2$ (with $k \geq 2$):

$$B_2^{h.c.}\left(k, l = \frac{1}{2}, T\right) = -\frac{\lambda_T^2}{8} - \frac{\lambda_T^2}{2(2l+1)^2}\left[[(\gamma_0+1) \mod 2-1]^2 + 3(\gamma_1^2 - 2\gamma_1)\right] \ .$$

It is $\omega_0 = -\frac{3}{2k}$ and $\omega_1 = \frac{1}{2k}$: since k is assumed to be ≥ 2, it follows $\gamma_0 = 2 - \frac{3}{2k}$, $(\gamma_0 + 1)\, mod\, 2 = 1 - \frac{3}{2k}$, and therefore $[(\gamma_0 + 1)\, mod\, 2 - 1]^2 = \frac{9}{4k^2}$, $\gamma_1 = \frac{1}{2k}$, hence

$$B_2^{h.c.}\left(k, l = \frac{1}{2}, T\right) = -\frac{\lambda_T^2}{8} - \frac{\lambda_T^2}{8}\left[\frac{9}{4k^2} + 3\left(\frac{1}{4k^2} - \frac{1}{k}\right)\right] = -\frac{\lambda_T^2}{8}\left(1 - \frac{3}{k} + \frac{3}{k^2}\right) \ .$$

$\star$ Case $l = 1$ (with $k \geq 4$):

$$B_2^{h.c.}(k, l = 1, T) = \frac{1}{(2l+1)^2}\sum_{j=0}^{2l}(2j+1)\left[\frac{1+(-1)^{j+2l}}{2} B_2^B(\omega_j, T) + \right.$$

$$\left.\frac{1-(-1)^{j+2l}}{2} B_2^F(\omega_j, T)\right] = \frac{1}{9}\left[B_2^B(\omega_0, T) + 3B_2^F(\omega_1, T) + 5B_2^B(\omega_2, T)\right] \ .$$

It is $\omega_0 = -\frac{4}{k}$, $\omega_1 = -\frac{2}{k}$, $\omega_2 = \frac{2}{k}$ and the ω_j's are such that $|\omega_j| \leq 1$: therefore, by using (5.29), one has

$$B_2^{h.c.}(k, l = 1, T) = \frac{\lambda_T^2}{36}\left[-1 + 4|\omega_0| - 2\omega_0^2 + 3(1 - 2\omega_1^2) + 5(-1 + 4|\omega_2| - 2\omega_2^2)\right] =$$

$$= \frac{\lambda_T^2}{36}\left[-1+4\cdot\frac{4}{k}-2\cdot\frac{16}{k^2}+3\left(1-2\cdot\frac{4}{k^2}\right)+5\left(-1+4\cdot\frac{2}{k}-2\cdot\frac{4}{k^2}\right)\right]=$$

$$= \lambda_T^2\left(-\frac{1}{12}+\frac{14}{9k}-\frac{8}{3k^2}\right) \quad .$$

$\star$ Case of the large-k limit, with $\lim_{l\to\infty,k\to\infty}\frac{l^2}{k}=a<\frac{1}{2}$: we limit ourselves to the case of even $2l$ (the opposite one is similar). We define j_{crit} as the maximum integer such that $\omega_j<0$, and $x_{crit}=\sqrt{2}l$: it can be verified that $|\omega_j|<1$ for all j. We have

$$B_2^{h.c.}(k,l,T)=\frac{1}{(2l+1)^2}\left[\sum_{j\ \mathrm{even}=0}^{2l}(2j{+}1)B_2^B(\omega_j,T)+\sum_{j\ \mathrm{odd}=1}^{2l-1}(2j{+}1)B_2^F(\omega_j,T)\right]=$$

$$=\frac{\lambda_T^2/4}{(2l+1)^2}\left[\sum_{j\ \mathrm{even}=0}^{2l}(2j+1)(-1+4|\omega_j|-2\omega_j^2)+\sum_{j\ \mathrm{odd}=1}^{2l-1}(2j+1)(1-2\omega_j^2)\right]=$$

$$=\frac{\lambda_T^2/4}{(2l+1)^2}\left[\sum_{j\ \mathrm{even}=0}^{j_{crit}}(2j{+}1)(-1{-}4\omega_j{-}2\omega_j^2)+\sum_{j\ \mathrm{even}=j_{crit}+1}^{2l}(2j{+}1)(-1{+}4\omega_j{-}2\omega_j^2)\right.$$

$$\left.+\sum_{j\ \mathrm{odd}=1}^{2l-1}(2j{+}1)(1{-}2\omega_j^2)\right]\simeq\frac{\lambda_T^2/4}{(2l+1)^2}\left[\frac{1}{2}\int_0^{x_{crit}}dx\,(2x{+}1)\left(-1{-}4\frac{x^2-2l^2}{k}{-}2\frac{(x^2-2l^2)^2}{k^2}\right)+\right.$$

$$\left.+\frac{1}{2}\int_{x_{crit}}^{2l}dx\,(2x{+}1)\left(-1{+}4\frac{x^2-2l^2}{k}{-}2\frac{(x^2-2l^2)^2}{k^2}\right){+}\frac{1}{2}\int_0^{2l}dx\,(2x{+}1)\left(1{-}2\frac{(x^2-2l^2)^2}{k^2}\right)\right]\simeq$$

$$\simeq\lambda_T^2\left(\frac{a}{2}-\frac{2}{3}a^2\right) \quad .$$

Comparison with Ref.[111]

With the notation used in the main text, Eq. (2) of Ref. [111] for NACS particles in the hard-core limit reads

$$B_2^{h.c.}(\kappa,l,T)=\frac{1}{(2l+1)^2}\sum_{j=0}^{2l}(2j{+}1)\left[\frac{1+(-1)^{j+2l}}{2}B_2^B(\omega_j,T)+\frac{1-(-1)^{j+2l}}{2}B_2^F(\omega_j,T)\right] \quad , \tag{5.49}$$

where $\omega_j \equiv \frac{1}{4\pi\kappa}\left[j(j+1)-2l(l+1)\right]$ and $B_2^{B,F}(\omega,T)$ is given by [91]

$$B_2^{B(F)}(\omega,T)=\frac{1}{4}\lambda_T^2\left\{\begin{array}{ll}-1+4\delta-2\delta^2, & \text{N even (odd)}\\ 1-2\delta^2, & \text{N odd (even)}\end{array}\right. \tag{5.50}$$

with $\omega = N+\delta$ and N an integer such that $0\leq\delta<1$. Eq. (5.49) can be derived as in the following: with the notation $\gamma_j \equiv \omega_j \, mod\, 2$, using Eqs. (4.34)-(4.36)-(4.38) one has

$$B_2^{h.c.}(\kappa,l,T)=B_2^{(n.i.)}(l,T)-\frac{2\lambda_T^2}{(2l+1)^2}\left[Z_2'(\kappa,l,T)-Z_2'^{(n.i.)}(l,T)\right]=$$

$$=-\frac{1}{4}\frac{\lambda_T^2}{(2l+1)^2}\sum_{j=0}^{2l}(2j+1)\left[\frac{1+(-1)^{j+2l}}{2}+\frac{1-(-1)^{j+2l}}{2}(-1)\right]+$$

$$-\frac{2\lambda_T^2}{(2l+1)^2}\sum_{j=0}^{2l}(2j+1)\left[\frac{1+(-1)^{j+2l}}{2}\frac{1}{4}(\gamma_j^2-2\gamma_j)+\frac{1-(-1)^{j+2l}}{2}\frac{1}{4}[(\gamma_j+1)\, mod\, 2-1]^2\right]=$$

$$=\frac{\lambda_T^2/4}{(2l+1)^2}\sum_{j=0}^{2l}(2j+1)\left[\frac{1+(-1)^{j+2l}}{2}(-1+4\gamma_j-2\gamma_j^2)+\right.$$

$$\left.+\frac{1-(-1)^{j+2l}}{2}(1-2[(\gamma_j+1)\, mod\, 2-1]^2)\right]=$$

$$=\frac{1}{4}\frac{\lambda_T^2}{(2l+1)^2}\sum_{j=0}^{2l}(2j+1)\left[\frac{1+(-1)^{j+2l}}{2}\left\{\begin{array}{ll}-1+4\delta_j-2\delta_j^2, & N_j \text{ even}\\ 1-2\delta_j^2, & N_j \text{ odd}\end{array}\right\}+\right.$$

$$\left.+\frac{1-(-1)^{j+2l}}{2}\left\{\begin{array}{ll}1-2\delta_j^2, & N_j \text{ even}\\ -1+4\delta_j-2\delta_j^2, & N_j \text{ odd}\end{array}\right\}\right]=$$

$$=\frac{1}{(2l+1)^2}\sum_{j=0}^{2l}(2j+1)\left[\frac{1+(-1)^{j+2l}}{2}B_2^B(\omega_j,T)+\frac{1-(-1)^{j+2l}}{2}B_2^F(\omega_j,T)\right] \ ,$$

that is nothing else than the (5.49) itself.

Notice that Eq. (3) of [111] should be replaced with

$$\frac{4}{\lambda_T^2} B_2^{h.c.}(\alpha, T) = \frac{1}{(2l+1)^2} \sum_{j=0}^{2l} (2j+1)[\mp(-1)^{j+2l}] + \frac{2}{(2l+1)^2} \sum_{j=0}^{2l} (2j+1)\delta_j [1 \pm (-1)^{j+2l} - \delta_j] \ , \tag{5.51}$$

where the upper and lower signs refer to the cases of even and odd N_j's: Eq. (5.51) is equivalent to our formula (4.38) and to Eq. (5.49).

We discuss now the limit $\alpha \equiv \frac{1}{4\pi\kappa} \to 0$: we observe that by a careful inspection it is possible to conclude that Eq. (30) of [110] and Eq. (3) of [111] do not tend in this limit to the correct value $B_2^{(n.i.)}(l, T)$ given in Eq. (4.35). However, the manipulation of the corrected version (5.51) above presented reproduces (as expected) the value $B_2^{h.c.} \to -\frac{\lambda_T^2}{4} \frac{1}{2l+1}$ for $\alpha \to 0$. Indeed for vanishing coupling constant α, using the same convention used above (upper and lower choices referring to the cases of even and odd N_j respectively), one has

$$\frac{1}{4\pi\kappa} \to 0 \Rightarrow \frac{1}{(2l+1)^2} \sum_{j=0}^{2l} (2j+1)[\mp(-1)^{j+2l}] + \frac{2}{(2l+1)^2} \sum_{j=0}^{2l} (2j+1)\delta_j [1 \pm (-1)^{j+2l} - \delta_j] \to$$

$$\frac{2}{(2l+1)^2} \sum_{j=0}^{2l} (2j+1)\left(\frac{1}{2} \mp \frac{1}{2}\right)\left[1 \pm (-1)^{j+2l} - \left(\frac{1}{2} \mp \frac{1}{2}\right)\right] + \frac{1}{(2l+1)^2} \sum_{j=0}^{2l} (2j+1)[\mp(-1)^{j+2l}] =$$

$$= -\frac{\sum_{j=0}^{2l} (2j+1)(-1)^{j+2l}}{(2l+1)^2} = -\frac{1}{2l+1} \ ,$$

as it might.

Spectrum in the soft-core non-Abelian case

In Subsection 4.3.2 we stated that the spectrum of the multi-component projected Hamiltonian operator H'_j can be represented in general as the union of the $(2j+1)$ spectra of the corresponding scalar Schrödinger operators. That follows from the following remark: the non-Abelian generalization of the soft-core expression (4.8) is Eq. (4.32). By denoting

$$A(r) \equiv \frac{1}{M} \left[-\frac{1}{r} \frac{d}{dr} r \frac{d}{dr} + \frac{(n+\alpha)^2}{r^2} \right] \ ,$$

the non-Abelian generalization of (4.6) is the $(2l+1)^2-$dimensional matricial equation

$$\begin{pmatrix} A(r) & 0 & 0 & \cdots \\ 0 & A(r) & 0 & \cdots \\ 0 & 0 & A(r) & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} \begin{pmatrix} R_0^{0,0} \\ R_0^{1,-1} \\ R_0^{1,0} \\ \cdots \end{pmatrix} = E \begin{pmatrix} R_0^{0,0} \\ R_0^{1,-1} \\ R_0^{1,0} \\ \cdots \end{pmatrix} = \frac{k^2}{M} \begin{pmatrix} R_0^{0,0} \\ R_0^{1,-1} \\ R_0^{1,0} \\ \cdots \end{pmatrix} . \tag{5.52}$$

The hard-disk regularization $R_0^{j,j_z}(r = R) = 0$ for all j, j_z discretizes the energy spectrum; let then $Sp_{j,j_z}(R)$ be the (discretized) spectrum of the component equation $A(r)R_0^{j,j_z} = ER_0^{j,j_z}$ restricted over the domain in which R_0^{j,j_z} has hard-core parameter ε_{j,j_z}, and $Sp(R)$ be the (discretized) spectrum of Eq. (5.52) in the domain in which any component R_0^{j',j'_z} has the respective assigned hard-core parameter ε_{j',j'_z}. If $E_{j',j'_z} \in Sp_{j',j'_z}(R)$ for some (j', j'_z), then also $E_{j',j'_z} \in Sp(R)$, because

$$\begin{pmatrix} A(r) & 0 & 0 & \cdots & \cdots \\ 0 & A(r) & 0 & \cdots & \cdots \\ 0 & 0 & A(r) & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \begin{pmatrix} 0 \\ \cdots \\ R_0^{j',j'_z} \\ 0 \\ \cdots \end{pmatrix} = E_{j',j'_z} \begin{pmatrix} 0 \\ \cdots \\ R_0^{j',j'_z} \\ 0 \\ \cdots \end{pmatrix} , \tag{5.53}$$

and all the null components of the vector trivially fulfill whichever hard-core conditions, in particular the assigned sequence $\{\varepsilon_{j,j_z}\} \in \{[0,\infty)\}^{(2l+1)^2}$. In conclusion, the spectrum in the non-Abelian case can be written as the above-mentioned union of spectra, which will automatically include all the possible relevant energy degenerations to be considered in the partition function for the computation of the virial coefficients.

Definition of the used Bessel functions

In the main text we used the Bessel functions of the first kind J_α and the modified Bessel function of the first kind I_α: their definition is respectively given by

$$J_\alpha(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m+\alpha}$$

and

$$I_\alpha(x) = i^{-\alpha} J_\alpha(ix) = \sum_{m=0}^{\infty} \frac{1}{m!\,\Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m+\alpha} . \tag{5.54}$$

The Lommel functions of two variables are defined in Eq.(5) of pg. 537 of [134] and read

$$U_\nu(w,z) = \sum_{m=0}^{\infty} (-1)^m \left(\frac{w}{z}\right)^{\nu+2m} J_{\nu+2m}(z)$$

$$V_n(w,z) = \cos\left(\frac{w}{2} + \frac{z^2}{2w} + \frac{\nu\pi}{2}\right) + U_{2-\nu}(w,z) \; .$$

Properties of the statistical potential v_α

In this Appendix we provide details and further informations on the statistical potential v_α for Abelian anyons.

We first give a derivation of Eq.(5.10): the relative PF (5.7) is given by

$$Z' = \frac{1}{2} \sum_{l=-\infty}^{\infty} \int_0^\infty dx\, e^{-x}\, I_{|l-\alpha|}(x), \quad x = Mr^2/2\beta\hbar^2 \tag{5.55}$$

and it can be rewritten as

$$Z' = \frac{1}{2h^2} \int d^2p\, e^{-\beta p^2/M} \int d^2r \sum_{l=-\infty}^{\infty} 2e^{-Mr^2/2\beta\hbar^2}\, I_{|l-\alpha|}\left(\frac{Mr^2}{2\beta\hbar^2}\right) , \tag{5.56}$$

as pointed out in [131]. That allows for its comparison with the PF (5.9) for classical systems associated to a generic potential $v(r)$, whence the result (5.10).

We also observe that

$$e^{-\beta v_\alpha(r)} = 2\, e^{-x} \left[i^{-\alpha} \sum_{n=0}^{\infty} (-1)^n J_{2n+\alpha}(ix) + i^{\alpha-2} \sum_{n=0}^{\infty} (-1)^n J_{2n+2-\alpha}(ix) \right] . \tag{5.57}$$

The expression that follows here below is a possible closed form for the statistical potential, but at the cost of using an integral representation given in formula (7), pg. 652 of [140]. It stands for any complex number μ in the vertical strip $-1 < \operatorname{Re}\mu < \alpha$:

$$e^{-\beta v_\alpha(r)} = 2\, e^{-x} \left[i^{-\alpha} \sum_{n=0}^{\infty} (-1)^n J_{2n+\alpha}(ix) + i^{\alpha-2} \sum_{n=0}^{\infty} (-1)^n J_{2n+2-\alpha}(ix) \right] =$$

$$= 2\, e^{-x} \left\{ i^{-\alpha} \left[\frac{1}{2} \int_0^{ix} J_\mu(ix-t)\, J_{\alpha-\mu-1}(t)\, dt \right] + i^{\alpha-2} \left[\frac{1}{2} \int_0^{ix} J_\mu(ix-t)\, J_{1-\alpha-\mu}(t)\, dt \right] \right\} =$$

$$= e^{-x}\left[i^{-\alpha}\int_0^{ix} J_\mu(ix-t)\; J_{\alpha-\mu-1}(t)\; dt - i^{\alpha}\int_0^{ix} J_\mu(ix-t)\; J_{1-\alpha-\mu}(t)\; dt\right] . \tag{5.58}$$

A simpler integral representation, valid for $\alpha \in (0,2)$, can be produced by using the following property ([134], pg. 540) of the bivariate Lommel function U:

$$U_\nu(w,z) = \frac{w^\nu}{z^{\nu-1}}\int_0^1 J_{\nu-1}(zt)\; \cos\left\{\frac{1}{2}w(1-t^2)\right\}\; t^\nu\, dt, \qquad \mathrm{Re}(\nu) > 0$$

together with expressions (5.57), (5.13) and (5.54). The resulting integral representation is

$$e^{-\beta v_\alpha(r)} = 2\,x\,e^{-x}\int_0^1 \cosh\left[\frac{x}{2}(1-t^2)\right]\left(I_{\alpha-1}(xt)\,t^\alpha \;+\; I_{1-\alpha}(xt)\,t^{2-\alpha}\right)\, dt\; . \tag{5.59}$$

In the final part of this Appendix we provide the derivation of the integral representation (5.14). To this end, we use the following representation [137] for the modified Bessel function of the first kind:

$$I_\nu = \frac{1}{\pi}\int_0^\pi e^{z\,\cos\theta}\,\cos\nu\theta\, d\theta - \frac{\sin\nu\pi}{\pi}\int_0^\infty e^{-z\,\cosh t - \nu t}\, dt, \quad \arg|z| \le \frac{\pi}{2},\ \mathrm{Re}\,\nu > 0\; . \tag{5.60}$$

Then the summation term in (5.10) is:

$$\sum_{n=-\infty}^{\infty} I_{|2n-\alpha|}(z) = \sum_{n=0}^{\infty} I_{2n+2-\alpha}(z) + \sum_{n=0}^{\infty} I_{2n+\alpha}(z) =$$

$$= \frac{1}{\pi}\int_0^\pi d\phi\; e^{z\cos\phi}\left[\sum_{n=0}^{\infty}\cos{(2n+\alpha)\phi} + \sum_{n=1}^{\infty}\cos{(2n-\alpha)\phi}\right] - \frac{1}{\pi}\int_0^\infty dt\; e^{-z\cosh t} f(t,\alpha)\; , \tag{5.61}$$

where

$$f(t,\alpha) = \sum_{n=0}^{\infty} e^{-(2n+\alpha)t}\sin{(2n+\alpha)\pi} + \sum_{n=1}^{\infty} e^{-(2n-\alpha)t}\sin{(2n-\alpha)\pi} = \sin\alpha\pi\frac{\sinh[(1-\alpha)t]}{\sinh t}\; ,$$

for $t \neq 0$ and

$$f(0,\alpha) \equiv \lim_{t\to 0^\pm} f(t,\alpha) = (1-\alpha)\sin\alpha\pi\; .$$

The first addend of the last integral representation is

$$\frac{1}{\pi}\int_0^\pi d\phi\; e^{z\cos\phi}\left[\sum_{n=0}^{\infty}\cos{(2n+\alpha)\phi} + \sum_{n=1}^{\infty}\cos{(2n-\alpha)\phi}\right] = \frac{1}{\pi}\int_0^\pi d\phi\; e^{z\cos\phi}\left[\sum_{n=0}^{\infty}\cos{(2n+\alpha)\phi} + \right.$$

$$+ \sum_{n=-\infty}^{-1} \cos{(2n+\alpha)\phi} \Bigg] = \frac{1}{\pi} \int_0^\pi d\phi \; e^{z\cos\phi} \sum_{n=-\infty}^{+\infty} \cos{(2n+\alpha)\phi} =$$

$$\frac{1}{\pi} \int_0^\pi d\phi \; e^{z\cos\phi} \; \mathrm{Re} \left[\sum_{n=-\infty}^{+\infty} (e^{i\alpha\phi} e^{2in\phi}) \right] =$$

$$= \frac{1}{\pi} \int_0^\pi d\phi \; e^{z\cos\phi} \; \mathrm{Re} \left[e^{i\alpha\phi} \, 2\pi \, \frac{\delta(2\phi) + \delta(2\phi - 2\pi)}{2} \right] = \frac{1}{2} \left(e^z + e^{-z} \; \cos\alpha\pi \right) \; . \tag{5.62}$$

As a result of (5.10), (5.61) and (5.62), one has then

$$e^{-\beta v_\alpha(r)} = 1 + e^{-2z} \cos\alpha\pi - 2 \, \frac{\sin\alpha\pi}{\pi} \, e^{-z} \int_0^\infty dt \, \frac{\sinh{[(1-\alpha)t]}}{\sinh t} \, e^{-z\,\cosh t} \; . \tag{5.63}$$

By direct inspection this result, notwithstanding the hypothesis of validity for (5.60), is valid also for α at the extremes of the interval $[0, 2]$, so that the derivation of (5.14) is completed.

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